First-Order Logic

Chapter 8
Problem of Propositional Logic

Propositional logic has very limited expressive power

– E.g., cannot say "pits cause breezes in adjacent squares" except by writing one sentence for each square.

– We want to be able to say this in one single sentence: "for all squares and pits, pits cause breezes in adjacent squares.

– First order logic will provide this flexibility.
First-order logic

• Propositional logic assumes the world contains **facts** that are true or false.

• First-order logic assumes the world contains
  – **Objects**: people, houses, numbers, colors, baseball games, wars, …
  – **Relations between objects**: red, round, prime, brother of, bigger than, part of, comes between, …
Relations

• Some relations are **properties**: they state some fact about a single object: Round(ball), Prime(7).

• **n-ary relations** state facts about two or more objects: Married(John,Mary), Largerthan(3,2).

• Some relations are **functions**: their value is another object: Plus(2,3), Father(Dan).
Models for FOL: Example
Atomic Sentences

- Sentences in logic state facts that are true or false.

- Properties and n-ary relations do just that:
  - LargerThan(2,3) (means 2>3) is false.
  - Brother(Mary,Pete) is false.

- Note: Functions do not state facts and form no sentence:
  - Brother(Pete) refers to the object John (his brother) and is neither true nor false.

- Brother(Pete,Brother(Pete)) is True.
Complex Sentences

• We make complex sentences with connectives (just like in proposition logic).

\[ \neg \text{Brother}(\text{LeftLeg}(\text{Richard}), \text{John}) \lor (\text{Democrat}(\text{Bush})) \]
Quantification

• Round(ball) is true or false because we give it a single argument (ball).

• We can be much more flexible if we allow variables which can take on values in a domain. e.g. reals x, all persons P, etc.

• To construct logical sentences we need a quantifier to make it true or false.
Quantifier

• Is the following true or false? \( x > 5, \ x \in \mathbb{R} \)

• To make it true or false we use \( \forall \) and \( \exists \)

\[
\forall x [(x > 2) \Rightarrow (x > 3)] \quad x \in \mathbb{R} \quad (\text{false})
\]

\[
\exists x [(x^2 = -1)] \quad x \in \mathbb{R} \quad (\text{false})
\]

For all real \( x, \ x > 2 \) implies \( x > 3 \). There exists some real \( x \) which square is minus 1.
Nested Quantifiers

• Combinations of universal and existential quantification are possible:

$$\forall x \forall y \text{Father}(x, y) \equiv \forall y \forall x \text{Father}(x, y)$$

$$\exists x \exists y \text{Father}(x, y) \equiv \exists y \exists x \text{Father}(x, y)$$

$$\forall x \exists y \text{Father}(x, y) \neq \exists y \forall x \text{Father}(x, y)$$

$$\exists x \forall y \text{Father}(x, y) \neq \forall y \exists x \text{Father}(x, y)$$

$$x, y \in \{\text{All people}\}$$

Quiz: which is which:
- Everyone is the father of someone.
- Everyone has everyone as a father.
- There is a person who has everyone as a father.
- There is a person who has a father.
- There is a person who is the father of everyone.
- Everyone has a father.

Binary relation: “x is a father of y.”
De Morgan’s Law for Quantifiers

De Morgan’s Rule

\[ P \land Q \equiv \neg(\neg P \lor \neg Q) \]
\[ P \lor Q \equiv \neg(\neg P \land \neg Q) \]
\[ \neg(P \land Q) \equiv \neg P \lor \neg Q \]
\[ \neg(P \lor Q) \equiv \neg P \land \neg Q \]

Generalized De Morgan’s Rule

\[ \forall x \ P \equiv \neg \exists x \ (\neg P) \]
\[ \exists x \ P \equiv \neg \forall x \ (\neg P) \]
\[ \neg \forall x \ P \equiv \exists x \ (\neg P) \]
\[ \neg \exists x \ P \equiv \forall x \ (\neg P) \]

Rule is simple: if you bring a negation inside a disjunction or a conjunction, always switch between them (or \( \rightarrow \) and, and \( \rightarrow \) or).

  This relates two objects.
Common mistakes to avoid

- $\Rightarrow$ is the main connective with $\forall$
- $\land$ is the main connective with $\exists$

$\forall x, \text{King}(x) \Rightarrow \text{Person}(x)$  $x = \{\text{Pete, Mary, tablespoon}\}$
$\forall x, \text{King}(x) \land \text{Person}(x)$
$\exists x, \text{King}(x) \Rightarrow \text{Person}(x)$
$\exists x, \text{King}(x) \land \text{Person}(x)$

All of these must be true!
King(Pete) AND Person(Pete)
King(Mary) AND Person(Mary)
King(Tablespoon) AND Person(Tablespoon)

One of these should be true!
if King(Pete) then Person(Pete)
if King(Mary) then Person(Mary)
If King(Tablespoon) then Person(Tablespoon) True!

too strong    False!

too weak
Using FOL

• We want to TELL things to the KB, e.g.
  TELL(KB, \( \forall x, \text{King}(x) \Rightarrow \text{Person}(x) \) )

• We also want to ASK things to the KB,
  ASK(KB, \( \exists x, \text{Person}(x) \) )

• The KB should return the list of x’s for which Person(x) is true:
  \{x/John, x/Richard, \ldots\}
Examples

The kinship domain:

- Brothers are siblings
  \[ \forall x,y \; \text{Brother}(x,y) \Rightarrow \text{Sibling}(x,y) \]

- One's mother is one's female parent
  \[ \forall m,c \; \text{Mother}(c) = m \Leftrightarrow (\text{Female}(m) \land \text{Parent}(m,c)) \]

- “Sibling” is symmetric
  \[ \forall x,y \; \text{Sibling}(x,y) \Leftrightarrow \text{Sibling}(y,x) \]

Some may be considered axioms, others as theorems which can be derived from the axioms.
Translating English to FOL

• Every gardener likes the sun.
  \((Ax) \ gardener(x) \Rightarrow likes(x, \text{Sun})\)

• You can fool some of the people all of the time.
  \((Ex)(At) \ (person(x) \land time(t)) \Rightarrow can-fool(x, t)\)

• You can fool all of the people some of the time.
  \((Ax)(Et) \ (person(x) \land time(t)) \Rightarrow can-fool(x, t)\)

• All purple mushrooms are poisonous.
  \((Ax) \ (mushroom(x) \land purple(x)) \Rightarrow poisonous(x)\)

• No purple mushroom is poisonous.
  \(\sim(Ex) \ purple(x) \land mushroom(x) \land poisonous(x)\)
  or, equivalently,
  \((Ax) \ (mushroom(x) \land purple(x)) \Rightarrow \sim\text{poisonous}(x)\)

• There are exactly two purple mushrooms.
  \((Ex)(Ey) \ mushroom(x) \land purple(x) \land mushroom(y) \land purple(y) \land \sim(x=y) \land (Az) \ (mushroom(z) \land purple(z)) \Rightarrow ((x=z) \lor (y=z))\)
1. (12 pts) **First Order Logic**

Consider the following sentence in FOL:

$$\forall x \text{ Married}(\text{Father}(x), \text{Mother}(x)) \Rightarrow$$

$$\exists y \text{ Certificate}(y) \land \text{Names}(y, \text{Father}(x), \text{Mother}(x))$$

In English: *For every person who has a father and a mother that are married, there exists a paper which is a wedding certificate and which contains the names of both the father and the mother of this person.*

a. (4 pts) Identify the functions, properties, binary relations, relations with arity 3, quantifiers and connectives in this sentence.

b. (4 pts) Consider the sentence: $$\exists x \ [p(x) \Rightarrow q(x)]$$. Assume we know that there is no value for $x$ for which $q(x)$ is true. Is it still possible that the above sentence is true?

c. (4 pts) Provide the truth-table for $$(P \lor Q) \Rightarrow (P \land Q)$$ where $P$ and $Q$ are two propositions.
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   In English: *For every person who has a father and a mother that are married, there exists a paper which is a wedding certificate and which contains the names of both the father and the mother of this person.*

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   a) answer: Function: Father, Mother, Property: Certificate, Binary Relation: Married, Tertiary Relation: Names, Quantifiers: \(\forall, \exists\), Connectives: \(\rightarrow, \land\).

   b. (4 pts) Consider the sentence: \(\exists x [p(x) \Rightarrow q(x)]\). Assume we know that there is no value for \(x\) for which \(q(x)\) is true. Is it still possible that the above sentence is true?

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b) answer: Yes, because $p(x)$ can still be false.

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c. (4 pts) Provide the truth-table for \((P \lor Q) \Rightarrow (P \land Q)\) where \(P\) and \(Q\) are two propositions.

b) answer: \(P:T,T,F,F, Q:T,F,T,F, (P \lor Q) \Rightarrow (P \land Q):T,F,F,T.\)
3. (10 pts) **First Order Logic**

Consider the following relations: $F(x)$ is true when $x$ is female, $M(x)$ is true when $x$ is male, $D(x)$ is true when $x$ lives in Disneyland and $L(x, y)$ is true when $x$ likes $y$. Translate the following sentences into first order logic:

a. (5 pts) There is at least one male and female, both living in Disneyland, that like each other.

b. (5 pts) All males and females living in Disneyland like each other.
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   a) answer: $\exists x, y \ M(x) \wedge F(y) \wedge D(x, y) \wedge L(x, y) \wedge L(y, x)$.

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b. (5 pts) All males and females living in Disneyland like each other.
   b) answer: $\forall x, y \ M(x) \land F(y) \land D(x, y) \Rightarrow L(x, y) \land L(y, x)$. 