1. (2 points) Truth Tables
   Let p and q be 2 propositions.
   a. Construct truth tables for the following compound propositions:
      (i) \((p \land \neg q) \rightarrow q\), (ii) \((p \lor q) \leftrightarrow (p \land q)\).
   answer: If we use the following ordering in the truth table:
   \(p: T T F F\), \(q: T F T F\), then answers are (i): T F T T, (ii) T F F T.
   b. Assume we have \(n\) propositions \(p_1, \ldots, p_n\), with \(n\) a positive integer and
      we construct a compound proposition which includes all \(n\) propositions above. How many entries
      would the truth table have for arbitrary \(n\) (i.e. how many different assignments can the string
      of propositions \(p_1, \ldots, p_n\) have)?
   answer: \(2^n\).

2. (4 points → 2 for each item) Sets and Logic
   Let \(U\) (the universal set) be equal to all integers \(Z\). Define the sets \(A, B, C\) as follows:
   \[
   A = \{ x | (x \in Z) \land (0 \leq x \leq 1) \} \\
   B = \{ x | (x \in Z) \land (x \text{ odd}) \} \\
   C = \{ x | (x \in Z) \land (x \text{ even}) \land (x > 0) \} 
   \]
   a. Determine the following sets (any set notation treated in class is acceptable in answering your
      questions):
      (i) \(A \cup C\), (ii) \((B \cup C) \cap C\), (iii) \(B\), (iv) \(A - B\).
   answer: (i) \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\), (ii) \(\{2\}\), (iii) \(\{2, 4, 6, 8\}\), (iv) \(\emptyset\).
   b. Are the following propositions true?
      \[
      \forall x [(x \in A) \land (x \in B)] \rightarrow (x > 0) \\
      \exists x [(x \in (A - B)) \leftrightarrow (x = 2)]
      \]
   answer: (i) T, (ii) T (e.g. \(x = 1\)).

3. (2 points+bonus) Bottles of wine
   A store sells three kinds of wine: red, white and port. Of each kind the store has 10 indistinguishable
   bottles.
a. 3 different customers arrive. Customer 1 buys 1 bottle, customer 2 buys 2 bottles, customers 3 buys 3 bottles. In how different many ways can the three customers buy their wine?

answer: $C(3, 1) \times C(4, 2) \times C(5, 3) = 3 \times 6 \times 10 = 180$

b. Next, 4 identical (indistinguishable) brothers walk into the store. Each brother buys 1 bottle. In how many ways can they buy their bottles of wine?

answer: $C(6, 4) = 15$

Bonus. Finally, 3 identical (indistinguishable) sisters walk into the store, each buying 2 bottles of wine. In how many ways can they buy their bottles of wine? (hint: imagine that pairs of bottles are sold in a bag. There are as many different bags as there are different pairs of bottles. Each sister buys 1 bag.)

answer: $C(8, 3) = 56$.

4. (4 points) The Biased Die

A gambler is confronted with a biased die. Let $X$ be the random variable which value is equal to the number of “eyes” on the die (“eyes” are markings/indentations on the die, the number of which represents the value of that side/face of the die).

a. The probability of $i$ eyes coming up is $P(X = i) = i \times p$, with $p$ a parameter. Compute the value of $p$?

answer: $p = 21$.

b. What is the expected value $E(X)$?

answer: $E(X) = 4\frac{1}{3}$.

c. What is the standard deviation $\sigma(X)$?

answer: $\sigma(X) = 1.4907$.

d. We throw the die 3 times. What is the probability of finding two 3’s and one 6, in any order?

answer: $3 \times 3^{2} \times 6/21^{3} = 0.0175$.

5. (3 points) Checking in Hats

At a party 10 people checking in their hats. As it turns out 5 of them are indistinguishable black hats, while the other 5 are distinguishable colored hats. The person at the counter gets utterly confused and forgets all about which hat belongs to who. In despair he decides to return them at random when people collect their hats.

a. What is the probability that someone who checked in a colored hat recollect the correct hat again?

answer: $\frac{1}{10}$.

b. Assuming that a person will not notice when he/she receives an indistinguishable hat from his own, what is the probability that a person who checked a black hat will go home with the “correct” hat?

answer: $\frac{1}{2}$.

c. Let $X_i$ be the random variable that takes value $X_i = 1$ if person $i$ receives the correct hat. Compute the expected number of correctly returned hats $E[X_1 + \ldots + X_{10}]$.

answer: $E[X_1 + \ldots + X_{10}] = 3$.

6. (4 points) Students and Grades

Consider the set $S$ of 30 students that currently take the ICS 6A class in discrete mathematics and the set $G$ of 4 possible grades $A, B, C, D$. 


a. What is the minimum number of students that will receive the same grade?

answer: Pigeonhole principle: 8.

b. Now consider a particular “grading”, where for each of the 4 grades, there is at least one student who receives that grade. Consider the function \( f : S \to G \), where \( f \) assigns the grades to the students. Is this function (i) one-to-one, (ii) onto, (iii) a one-to-one correspondence?

answer: (i) no, (ii) yes, (iii) no.

c. In how many ways can we assign grades to students?

answer: \( 4^{30} \).

d. Next, we assume each student receives a grade independently of the other students and that the probabilities of getting a particular grade are equal (i.e. \( 1/4 \)). Consider a subset of the students consisting of Joe, Fritz and George. Let \( E \) be the event that among those 3 students, exactly 2 receive an A and let \( F \) be the event that Fritz receives an A. Compute the following probabilities, \( P(E) \), \( P(E|F) \). Are the events \( E \) and \( F \) independent?

answer: \( P(E) = \frac{9}{64} \), \( P(E|F) = P(E \cap F)/P(F) = \frac{3}{32}/\frac{1}{4} = \frac{3}{8} \).

The events \( E \) and \( F \) are not independent.