Final Exam: Tu. June 14 2005, 4-6pm

Discrete Mathematics ICS 6A
Instructor: Max Welling

- *This exam is closed book*
- *Spend your time wisely: get a shot at each question.*
- *Write down your calculation and provide insights.*
- *If you didn't bring your calculator, you may simplify your answer as much as possible and leave it at that.*
- *Good Luck !*

1. (1+1+3+2=7 points) **Logic, Proofs, Sets**

   a. Prove the following equivalence using truth tables: $p \to q \equiv \neg p \vee q$

   A: $p = (T,T,F,F), q = (T,F,T,F), p \to q = (T,F,T,T), \neg p = (F,F,T,T), \neg p \vee q = (T,F,T,T)$.

   b. Is the following conclusion correct:
   If an apple is not red then it is not ripe.
   This apple is ripe.
   Therefore, this apple must be red.

   A: True: ($\neg$red $\to$ $\neg$ripe) $\to$ (ripe $\to$ red).

   c. There are $10,000$ republicans in village X, $3,000$ of which have blue eyes while $5,000$ republicans are older than 60 years of age. Moreover, there are $1,000$ republicans with blue eyes that are older than 60 years of age.
   i) What is the number of republicans that are older than 60 years of age or have blue eyes (or both)?
   ii) What is the number of republicans that are older than 60 years of age or have blue eyes, but not both?
   iii) What is the number of republicans that is younger than or equal to 60 years of age and do not have blue eyes.

   A: B=rep. with blue eyes. A=rep. older than 60, R=all rep.
   i) We want $|B \cup A| = |B| + |A| - |B \cap A| = 3,000 + 5,000 - 1,000 = 7,000$.
   ii) We want $|B \cup A| - |B \cap A| = |B| + |A| - |B \cap A| - |B \cap A| = 3,000 + 5,000 - 2 \times 1,000 = 6,000$.
   iii) $|\overline{B} \cap \overline{A}| = |\overline{B \cup A}| = |R| - (|B| + |A| - |B \cap A|) = 10,000 - 7,000 = 3,000$

   d. Determine for both statements below if they are propositions and determine the truth value for the ones which are propositions. The domain for all the variables is $\mathbb{R}$.
   i)$\exists x \; \forall y \; [x + y + z = 0]$
   ii)$\forall x \; \forall y \; [(x > y) \to (x^2 > y^2)]$

   A: i) Not a prop. ii) false (e.g. $x = -1, y = -2$).

2. (2+2=4 points) **Induction**

   a. Give an *indirect* proof of the theorem: "If $5n + 4$ is odd, then n is odd".

   A: We need to prove: "if n is even, then $5n + 4$ is even. Write: $n = 2k$ with k integer. Then $5n + 4 = 5(2k) + 4 = 2(5k + 2)$ where $5k + 2$ is integer, hence $5n + 4$ is even.

   b. Prove by mathematical induction that: $2^n > n^2$ for $n > 4$.

   A: Base: $n = 5$: $2^5 = 32$ and $5^2 = 25$. Assume the theorem is true for k, then we need to prove: $2^{k+1} > (k + 1)^2$. Left hand side is $2(2)^k = 2^k + 2^k$. Right hand side is: $k^2 + 2k + 1$. Since we know that $2^k > k^2$ we only need to prove that $2^k > 2k + 1$. This you could do by induction again or by using $2^k > k^2$ you only need to prove $k^2 > 2k + 1$. One way is to write: $k^2 > 2k+1 > 2k-1 \rightarrow k^2 - 2k + 1 > 0 \rightarrow (k-1)^2 > 0$ which is obvious because it is a square.

3. (2+2+2=6 points) **Counting**

   a. A class has 17 Hispanic and 19 African American students. We want to form a committee consisting of 8 students and each race must have at least 3 representatives. In how many ways can we form the committee?

   A: $C(17, 3)C(19, 5) + C(17, 4)C(19, 4) + C(17, 5)C(19, 3)$.

   b. A gambler wants to calculate in how many ways he can get "four of a kind" with a hand of five cards (e.g. 4 aces and some other card). There are 52 cards, 4 suites and 13 kinds in a deck of cards.

   A: 13×48. First 13 choices for a "kind" and then an arbitrary card from the remaining 48 cards.

   c. How many solutions are there to the problem: $x_1 + x_2 + ... + x_{13} = 17$ if $\{x_i\}$ are non-negative integers and we know that $x_1 > 0$.

   A: $C(16+13-1, 16)$ (put 17 indistinguishable balls into 13 buckets, where bucket 1 already contains one of the balls).

4. (1+1+1+2=5 points) **Probability I**
   A gambler plays the following game: A machine randomly generates a black or a red card, each with probability 1/2. He can see the entire history of cards that have been drawn. At any time he can decide to bet some money and predict the color of the next card. If he predicts correct he will get twice the amount in the bet, if he gets it wrong, he loses all his money in the bet. This gambler uses the following strategy: He waits until black has been drawn 5 times. He reasons: "since 6 times the same color is very unlikely it must be very likely that the color will change in round 6". So he puts his money on red in round 6.

   a. What is the probability that black is drawn 5 times in a sequence of 5?

   A: $1/2^5$

   b. What is the conditional probability that black is drawn in the sixth round, given the fact that black has been drawn 5 times in the first 5 rounds.

   A: 1/2

   c. Let A be the event that black is drawn 5 times in the first 5 rounds and let B be the event that black is drawn in the sixth round. Are A and B independent?

A: Yes, independent.

d. Let X be the random variable that counts the number of black cards in a total of 20 rounds. Compute the variance of X.

A: $20 \times 1/2 \times 1/2$

5. (1+2+2=5) points) **Probability II**
We distribute 20 different (distinguishable) books over 10 kids. Every time we give away another book we randomly pick one of the 10 kids.

a. How many ways are there to distribute the books (i.e. what is the cardinality of the sample space)?

A: 10^{20}.

b. What is the probability that all the books end up being given to one kid (any kid), leaving 9 kids with no books?

A: $10/10^{20}$ (it could happen to any of the 10 kids).

c. What is the probability that Mary (one of the 10 kids) ends up with no books at all?

A: At each round Mary has a probability of 9/10 to not get a book. Hence, over 10 independent trials she has a prob of $(9/10)^{20}$ to end up with nothing.

6. (2+2=4 points) **Recurrence Relations**

a. A researcher finds that a rabbit population follows the following recurrence relation: $a_t = 2a_{t-1}$. At $t = 0$ there are 2 rabbits.
Compute a general expression for the number of rabbits after T years, i.e. solve the recurrence relation.

A: $a_T = 2(2)^T$

b. A new researcher finds that the equation is actually slightly more complicated namely $a_t = 4a_{t-1} - 4a_{t-2}$. At $t = 0$ there are 2 rabbits and at $t = 1$ there are 8 rabbits.
Also solve this recurrence relation to find an expression for the number of rabbits after T years.

A: $a_T = 2(1 + T)(2)^T$