1. (8pts) **Sets and Logic**

Let \( U \) (the universal set) be equal to all integers \( \mathbb{Z} \). Define the sets \( A, B, C, D \) as follows:

\[
A = \{x | (x \in \mathbb{Z}) \land (x < 1)\}
\]

\[
B = \{x | (x \in \mathbb{Z}) \land (x > 9)\}
\]

\[
C = \{x | (x = 2k) \land (k \in \mathbb{Z})\}
\]

\[
D = \{x | (x \in \mathbb{Z}) \land (x > 12) \land (x < 5)\}
\]

a. (4pts) Determine the following sets (any set notation treated in class is acceptable in answering your questions):

(i) \( B \cup C \), (ii) \((A \cup C) \cap C\), (iii) \( D \), (iv) \( A - C \).

**answer:**

(i) \( \{x | (x \in \mathbb{Z}) \land ((x > 9) \lor (x = \text{even}))\} \)

(ii) \( \mathbb{Z} \)

(iii) \( \{x | (x \in \mathbb{Z}) \land (x < 1) \land (x = \text{odd})\} \)

b. (1pt) What is the powerset of \( \{\emptyset\} \)?

**answer:** \( \emptyset, \{\emptyset\} \)

c. (1pt) Give the truth table for \((p \land q) \rightarrow q\), where \( p \) and \( q \) are two propositions.

**answer:** \( T, T, T, T \) (a tautology)

d. (1pt) Is the following proposition a tautology \((p \lor q) \rightarrow p\)?

**answer:** NO (construct truth table and observe that it can be false)

e. (1pt) Determine the truth value of the following proposition: \(((F \leftrightarrow F) \land (T \leftrightarrow T)) \rightarrow F\) (\( F = \text{false}, T = \text{true} \)).

**answer:** \( F \)

2. (5pts) **Counting M&Ms**

Geoff buys one bag of M&Ms in a store. The bag contains 5 colors of M&Ms, with 10 M&Ms per color. M&Ms of the same color are indistinguishable.

a. (1pt) Mary arrives and she likes to have 7 M&Ms. In how many ways can Geoff give Mary 7 M&Ms if the order does not matter?

**answer:** \( C(5 + 7 - 1, 7) \)

b. (1pt) In how many ways can Geoff give Mary 7 M&Ms if the order does matter?
Geoff decides to mark each M&M by making small scratches on them. From now on all 50 M&Ms are distinguishable.

d.(1pt) In how many ways can Geoff give 7 M&Ms to Mary of the order does not matter?

\[
\text{answer: } C(50, 7)
\]

d.(1pt) In how many ways can Geoff give 7 M&Ms to Mary of the order does matter?

\[
\text{answer: } P(50, 7)
\]

e.(1pt) Finally, Geoff decides to take all his M&Ms back and put them into 5 distinguishable boxes, 10 each. In how many ways can Geoff do that (all M&Ms are still distinguishable)?

\[
\text{answer: } \frac{50!}{(10!)^5}
\]

3.(7pts) Probability

A gambler tosses a coin 12 times. The coin is biased and has a probability of \( p = \frac{3}{4} \) to come up heads. The order in which he tosses the coin matters.

a.(1pt) How many possible sequences can the gambler toss?

\[
\text{answer: } 2^{12}
\]

b.(1pt) What is the probability of tossing exactly 12 heads?

\[
\text{answer: } \left(\frac{3}{4}\right)^{12}
\]

c.(1pt) What is the probability of first tossing 6 heads and then 6 tails?

\[
\text{answer: } \left(\frac{3}{4}\right)^{6}\left(\frac{1}{4}\right)^{6}
\]

d.(1pt) What is the probability of tossing a total of 6 heads (in any position)?

\[
\text{answer: } C(12, 6)\left(\frac{3}{4}\right)^{6}\left(\frac{1}{4}\right)^{6}
\]

e.(1pt) What is the probability that the first head arrives at toss number 8?

\[
\text{answer: } \left(\frac{1}{4}\right)^{6}\left(\frac{3}{4}\right)
\]

Let \( X \) be a random variable counting the total number of heads in 12 tosses.

f.(1pt) What is the expected value of \( X \), i.e. \( E[X] \)?

\[
\text{answer: } 12\left(\frac{3}{4}\right)
\]

g.(1pt) Compute the standard deviation of \( X \), i.e. \( STD(X) \).

\[
\text{answer: } \sqrt{12\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)}
\]

4.(2pts) Counting

a.(1pt) How many ways are there to create new words from the word “IRVINE” by permuting the letters around?

\[
\text{answer: } 6!/2!
\]

b.(1pt) How many new words can we create if a word must contain the block “NE”?

\[
\text{answer: } 5!/2!
\]