This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a “fundamental theorem” you must indicate this** and explain why the theorem may be applied.

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.
1. Consider the following unconstrained optimization problem

\[
\text{minimize} \quad f(x) = \frac{1}{2} x^T P x + q^T x + c
\]

where \( x \in \mathbb{R}^n \), \( q \in \mathbb{R}^n \), and \( P \in S_{++}^n \) is an \( n \times n \) positive definite matrix. Write down the maximum number of steps that each of the following algorithms might take, and describe why.

(a) (5 points) Steepest gradient descent.

(b) (5 points) Newton’s method.

(c) (5 points) Conjugate gradient method.
2. Which of the following sets are convex, and why?

(a) (5 points) The set of points closer to a given point than a given set, i.e.,
\[ \{ x \in \mathbb{R}^n \mid \| x - x_0 \|_2 \leq \| x - y \|_2, \forall y \in S \} \]
where $S \subseteq \mathbb{R}^n$.

(b) (5 points) $S = \{ x \in \mathbb{R}^n \mid | \sum_{k=1}^{m} x_k \cos(kt) | \leq 1, \text{ for all } -\pi/3 \leq t \leq \pi/3 \}$, where $x_k$ is the $k$-th component of $x$.

(c) (5 points) The hyperbolic cone
\[ S = \{ x \in \mathbb{R}^n \mid x^T P x \leq (c^T x)^2, c^T x \geq 0 \}, \]
for a fixed vector $c$ and positive definite matrix $P$. 
3. For each of the following functions determine whether it is convex or concave.

(a) (5 points) \( f : R^m \to R \) with \( f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x) \).

(b) (5 points) \( f : S^n \to R \) with \( f(X) = \sup_{\|u\|_2 = 1} u^T X u \), where \( S^n \) denotes the set of symmetric real matrices. This function returns the largest eigenvalue of matrix \( X \).

(c) (5 points) \( f : S^+_n \to R \) with \( f(X) = \log \det(X) \) where \( S^+_n \) denotes the set of positive definite symmetric real matrices.
4. Consider the equality constrained entropy maximization problem

\[
(P_1) \quad \text{minimize} \quad f(x) = \sum_{i=1}^{n} x_i \log x_i
\]

subject to \(Ax = b\)

with \(\text{dom } f = \mathbb{R}^n_+\) (i.e., \(n\)-dim positive real numbers) and \(A \in \mathbb{R}^{p \times n}\). We assume the problem is feasible and that rank \(A = p < n\).

(a) (5 points) Write down the dual problem.

(b) (5 points) Write down the KKT conditions.

(c) (5 points) Derive the Newton’s step for solving the primal problem with a feasible start point.
5. Consider the linear program in standard form (with the optimization variable \( x \in \mathbb{R}^n \)):

\[
(P_1) \quad \text{minimize} \quad f_0(x) = c^T x \\
\text{subject to} \quad Ax = b \\
x \geq 0
\]

and the corresponding approximation using the logarithm barrier to eliminate the inequality constraint:

\[
(P_2) \quad \text{minimize} \quad tc^T x - \sum_{i=1}^{n} \log x_i \\
\text{subject to} \quad Ax = b
\]

(a) (10 points) Write down the KKT conditions associated with problems \( P_1 \) and \( P_2 \) and point out the differences.

(b) (5 points) Denote the optimal solutions of \( P_1 \) and \( P_2 \) by \( x^* \) and \( \bar{x}^* \) respectively. What can we say about the difference of the objective function at these two values: \( f_0(\bar{x}^*) - f_0(x^*) \)?

(c) (5 points) Derive the Newton’s direction for solving \( P_2 \) with a feasible start point.
6. In many applications, it is of interest to find a sparse “eigenvector” of a matrix $\Gamma \succeq 0$. That is, we would like to solve the following optimization problem, with optimization vector $x \in \mathbb{R}^n$:

$$
(P_1) \quad \text{maximize} \quad x^T \Gamma x \\
\text{subject to} \quad \|x\|_2 = 1 \\
\|x\|_1 \leq C
$$

where the inequality constraint with the $\ell_1$ norm (i.e., the $p$-norm with $p = 1$) regularizes the sparsity of the vector $x$.

(a) (5 points) Is the optimization problem $(P_1)$ an instance of a convex optimization? Why or why not?

(b) (5 points) Consider the semidefinite program with matrix variable $X \in S^n$:

$$
(P_2) \quad \text{maximize} \quad \text{tr}(\Gamma X) \\
\text{subject to} \quad X \succeq 0 \\
\text{tr}(X) = 1 \\
\sum_{i=1}^n \sum_{j=1}^n |X_{ij}| \leq C^2
$$

Denote the optimal value of $(P_1)$ by $a^*$ and the optimal value of $(P_2)$ by $b^*$. What is the relation between the two optimal values $a^*$ and $b^*$?
(c) (5 points) Compute the Lagrangian dual of problem $(P_2)$. (Introduce Lagrangian variables for all three constraints and find the dual function in terms of these variables.)

(d) (5 points) Set-up and describe the steps involved in implementing a barrier method to solve the dual of $(P_2)$. (Define a barrier function to remove the inequality constraints, and find the corresponding Newton’s direction.)