1. NLA 6.1
   SOLUTION:
   
   \[(I - 2P)(I - 2P) = I - 4P + 4P^2 = I\]

2. NLA 6.4
   SOLUTION: Given a matrix \(A\), the orthogonal projector onto the range of \(A\) can be expressed by the formula:
   
   \[P = A(A^*A)^{-1}A^*\]
   
   Using this formula, we have:
   
   (a) The orthogonal projector onto range(A) is
   
   \[
P = \begin{bmatrix}
   1/2 & 0 & 1/2 \\
   0 & 1 & 0 \\
   1/2 & 0 & 1/2
   \end{bmatrix}
   \]

   The image under \(P\) of the vector \((1, 2, 3)^*\) is \((2, 2, 2)^*\).

   (b) The orthogonal projector onto range(B) is
   
   \[
P = \begin{bmatrix}
   5/6 & 1/3 & 1/6 \\
   1/3 & 1/3 & -1/3 \\
   1/6 & -1/3 & 5/6
   \end{bmatrix}
   \]

   The image under \(P\) of the vector \((1, 2, 3)^*\) is \((2, 0, 2)^*\).

3. NLA 7.1
   SOLUTION:
(a) The QR factorization of $A$ is

$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(a) The QR factorization of $B$ is

$$B = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{\sqrt{6}} \\ 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{\sqrt{6}} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{6}}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

4. Determine (on paper) classical and modified Gram-Schmidt orthogonalization for the vectors

$$a_1 = (1, \epsilon, 0, 0)^T, \quad a_2 = (1, 0, \epsilon, 0)^T, \quad a_3 = (1, 0, 0, \epsilon)^T$$

During your calculation, make the approximation $1 + \epsilon^2 \approx 1$.

SOLUTION:

Assuming $\epsilon \to 0$:

(a) Classical GS: The resulting $Q$ matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ \epsilon & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

(b) Modified GS: The resulting $Q$ matrix is:

$$\begin{bmatrix} 1 & 0 & 0 \\ \epsilon & -1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

Check orthogonality:

(a) Classical GS: $<q_2, q_3> = 1/2$

(b) Modified GS: $<q_2, q_3> = 0$
5. NLA 7.3
SOLUTION:
Based on full QR factorization of $A$, $|\text{det}(A)| = |\text{det}(Q)\text{det}(R)| = |\text{det}(R)| = \prod_{j=1}^{m} r_{jj}$.

In addition, we note that $a_j = \sum_{i=1}^{j} r_{ij}q_i$. Therefore, $\|a_j\|_2 \geq r_{jj}$, by triangular inequality, for all $j = 1, \ldots, m$. And consequently, we have $|\text{det}(A)| = \prod_{j=1}^{m} r_{jj} \leq \prod_{j=1}^{m} \|a_j\|_2$.

6. NLA 8.2

7. Apply the $[Q, R] = \text{mgs}(A)$ function you have written in the previous problem to the following matrix

$$A = \begin{bmatrix} 0.70000 & 0.70711 \\ 0.70001 & 0.70711 \end{bmatrix}$$

Check the orthogonality of $Q$ matrix by calculating $\text{norm}(Q' \ast Q - \text{eye}(2))$. Compare the value returned by $\text{mgs}$ vs the one returned by the $\text{qr}$ function in Matlab.

8. NLA 10.1

SOLUTION: The householder reflector $F = I - 2qq^*$ with $q = v/\|v\|_2 \in \mathbb{C}^m$. The eigenvalues of $F$ are: 1 with multiplicity $m - 1$ - the corresponding eigenspace is the hyperplane orthogonal to $q$, and $-1$ with eigenvector $q$. The determinant of $F$ is -1. $F$ has $m$ singular values, all being 1.

9. NLA 10.2

10. NLA 10.3