CS295: Convex Optimization

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Convex set

Definition
A set $C$ is called **convex** if

$$x, y \in C \implies \theta x + (1 - \theta) y \in C \quad \forall \theta \in [0, 1]$$

In other words, a set $C$ is convex if the line segment between any two points in $C$ lies in $C$. 
Convex set: examples

Figure: Examples of convex and nonconvex sets
Convex combination

Definition
A convex combination of the points \( x_1, \ldots, x_k \) is a point of the form

\[
\theta_1 x_1 + \cdots + \theta_k x_k,
\]

where \( \theta_1 + \cdots + \theta_k = 1 \) and \( \theta_i \geq 0 \) for all \( i = 1, \ldots, k \).

A set is convex if and only if it contains every convex combinations of the its points.
Convex hull

Definition

The **convex hull** of a set $C$, denoted $\text{conv } C$, is the set of all convex combinations of points in $C$:

$$\text{conv } C = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in C, \theta_i \geq 0, i = 1, \ldots, k, \sum_{i=1}^{k} \theta_k = 1 \right\}$$
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$$

Properties:
- A convex hull is always convex
- $\text{conv } C$ is the smallest convex set that contains $C$, i.e., $B \supseteq C$ is convex $\implies \text{conv } C \subseteq B$
Convex hull: examples

Figure: Examples of convex hulls
Convex cone

A set $C$ is called a **cone** if $x \in C \implies \theta x \in C$, $\forall \theta \geq 0$. 
Convex cone

A set \( C \) is called a **cone** if \( x \in C \implies \theta x \in C, \ \forall \theta \geq 0 \).

A set \( C \) is a **convex cone** if it is convex and a cone, i.e.,

\[
x_1, x_2 \in C \implies \theta_1 x_1 + \theta_2 x_2 \in C, \ \forall \theta_1, \theta_2 \geq 0
\]
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The point $\sum_{i=1}^{k} \theta_i x_i$, where $\theta_i \geq 0, \forall i = 1, \cdots, k$, is called a **conic combination** of $x_1, \cdots, x_k$. 

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The **conic hull** of a set $C$ is the set of all conic combinations of points in $C$. 
Conic hull: examples

Figure: Examples of conic hull
A **hyperplane** is a set of the form \( \{ x \in \mathbb{R}^n \mid a^T x = b \} \) where \( a \neq 0, b \in \mathbb{R} \).
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A (closed) **halfspace** is a set of the form \( \{ x \in \mathbb{R}^n \mid a^T x \leq b \} \) where \( a \neq 0, b \in \mathbb{R} \).

- **a** is the normal vector
- hyperplanes and halfspaces are convex
Euclidean balls and ellipsoids

**Euclidean ball** in $\mathbb{R}^n$ with center $x_c$ and radius $r$:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$
Euclidean balls and ellipsoids

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ellipsoid in $\mathbb{R}^n$ with center $x_c$:

$$\mathcal{E} = \left\{ x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1 \right\}$$

where $P \in S^n_{++}$ (i.e., symmetric and positive definite)

- the lengths of the semi-axes of $\mathcal{E}$ are given by $\sqrt{\lambda_i}$, where $\lambda_i$ are the eigenvalues of $P$.
- An alternative representation of an ellipsoid: with $A = P^{1/2}$

$$\mathcal{E} = \{x_c + Au \mid \|u\|_2 \leq 1\}$$
Euclidean balls and ellipsoids

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▶ An alternative representation of an ellipsoid: with $A = P^{1/2}$

$$\mathcal{E} = \{ x_c + Au \mid \|u\|_2 \leq 1 \}$$

Euclidean balls and ellipsoids are convex.
Norms

A function $f : R^n \rightarrow R$ is called a **norm**, denoted $\|x\|$, if

- nonegative: $f(x) \geq 0$, for all $x \in R^n$
- definite: $f(x) = 0$ only if $x = 0$
- homogeneous: $f(tx) = |t|f(x)$, for all $x \in R^n$ and $t \in R$
- satisfies the triangle inequality: $f(x + y) \leq f(x) + f(y)$

**notation:** $\| \cdot \|$ denotes a general norm; $\| \cdot \|_{\text{symb}}$ denotes a specific norm

**Distance:** $dist(x, y) = \|x - y\|$ between $x, y \in R^n$. 

Examples of norms

- $\ell_p$-norm on $\mathbb{R}^n$: $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$
  - $\ell_1$-norm: $\|x\|_1 = \sum_i |x_i|$  
  - $\ell_\infty$-norm: $\|x\|_\infty = \max_i |x_i|$  

- Quadratic norms: For $P \in S_{++}^n$, define the $P$-quadratic norm as  
  \[ \|x\|_P = (x^T P x)^{1/2} = \|P^{1/2} x\|_2 \]
Let $\| \cdot \|_a$ and $\| \cdot \|_b$ be norms on $\mathbb{R}^n$. Then $\exists \alpha, \beta > 0$ such that
\[ \forall x \in \mathbb{R}^n, \quad \alpha \| x \|_a \leq \| x \|_b \leq \beta \| x \|_a. \]

Norms on any finite-dimensional vector space are equivalent (define the same set of open subsets, the same set of convergent sequences, etc.)
Dual norm

Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\| \cdot \|^*$, is defined as

$$\|z\|^* = \sup \{ z^T x \mid \|x\| \leq 1 \}.$$
Dual norm

Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\| \cdot \|_*$, is defined as

$$\|z\|_* = \sup \{ z^T x \mid \|x\| \le 1 \}.$$ 

- $z^T x \le \|x\| \|z\|_*$ for all $x, z \in \mathbb{R}^n$
- $\|x\|_{**} = \|x\|$ for all $x \in \mathbb{R}^n$
- The dual of the Euclidean norm is the Euclidean norm (Cauchy-Schwartz inequality)
Dual norm

Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$. The associated dual norm, denoted $\| \cdot \|_*$, is defined as

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- $z^T x \leq \|x\| \|z\|_*$ for all $x, z \in \mathbb{R}^n$
- $\|x\|_{**} = \|x\|$ for all $x \in \mathbb{R}^n$
- The dual of the Euclidean norm is the Euclidean norm (Cauchy-Schwartz inequality)
- The dual of the $\ell_p$-norm is the $\ell_q$-norm, where $1/p + 1/q = 1$ (Holder’s inequality)
- The dual of the $\ell_\infty$ norm is the $\ell_1$ norm
- The dual of the $\ell_2$-norm on $\mathbb{R}^{m \times n}$ is the nuclear norm,

$$\|Z\|_{2*} = \operatorname{sup} \{ \operatorname{tr}(Z^T X) \mid \|X\|_2 \leq 1 \} = \sigma_1(Z) + \cdots + \sigma_r(Z) = \operatorname{tr}(Z^T Z)^{1/2},$$

where $r = \text{rank } Z$. 
Norm balls and norm cones

**norm ball** with center $x_c$ and radius $r$: $\{x \mid \|x - x_c\| \leq r\}$

**norm cone**: $C = \{(x, t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$

- the second-order cone is the norm cone for the Euclidean norm

norm balls and cones are convex
A **polyhedron** is defined as the solution set of a finite number of linear equalities and inequalities:

\[
P = \{ x \mid Ax \preceq b, Cx = d \}
\]

where \( A \in \mathbb{R}^{m \times n}, C \in \mathbb{R}^{p \times n}, \) and \( \preceq \) denotes **vector inequality** or **componentwise inequality**.

A polyhedron is the intersection of finite number of halfspaces and hyperplanes.
Simplexes

The **simplex** determined by \( k + 1 \) affinely independent points \( \nu_0, \cdots, \nu_k \in \mathbb{R}^n \) is

\[
C = \text{conv}\{ \nu_0, \cdots, \nu_k \} = \left\{ \theta_0 \nu_0 + \cdots + \theta_k \nu_k \mid \theta \succeq 0, \mathbf{1}^T \theta = 1 \right\}
\]

The affine dimension of this simplex is \( k \), so it is often called \( k \)-dimensional simplex in \( \mathbb{R}^n \).

Some common simplexes: let \( e_1, \cdots, e_n \) be the unit vectors in \( \mathbb{R}^n \).

- **unit simplex**: \( \text{conv}\{0, e_1, \cdots, e_n\} = \{x \mid x \succeq 0, \mathbf{1}^T \theta \leq 1\} \)
- **probability simplex**: \( \text{conv}\{e_1, \cdots, e_n\} = \{x \mid x \succeq 0, \mathbf{1}^T \theta = 1\} \)
Positive semidefinite cone

notation:

- $S^n$: the set of symmetric $n \times n$ matrices
- $S^+_n = \{X \in S^n \mid X \succeq 0\}$: symmetric positive semidefinite matrices
- $S^{++}_n = \{X \in S^n \mid X \succ 0\}$ symmetric positive definite matrices

$S^+_n$ is a convex cone, called positive semidefinite cone. $S^{++}_n$ comprise the cone interior; all singular positive semidefinite matrices reside on the cone boundary.
Positive semidefinite cone: example

\[ X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S^2_+ \iff x \geq 0, z \geq 0, xz \geq y^2 \]

**Figure:** Positive semidefinite cone: \( S^2_+ \)
Operations that preserve complexity

- intersection
- affine function
- perspective function
- linear-fractional functions
Intersection

If $S_1$ and $S_2$ are convex, then $S_1 \cap S_2$ is convex.
Intersection

If $S_1$ and $S_2$ are convex. then $S_1 \cap S_2$ is convex.

If $S_\alpha$ is convex for every $\alpha \in \mathcal{A}$, then $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex.
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If $S_\alpha$ is convex for every $\alpha \in \mathcal{A}$, then $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex.
Show that the positive semidefinite cone $S^n_+$ is convex.

**Proof.**

$S^n_+$ can be expressed as

$$S^n_+ = \bigcap_{z \neq 0} \left\{ X \in S^n \mid z^T X z \geq 0 \right\}.$$ 

Since the set

$$\left\{ X \in S^n \mid z^T X z \geq 0 \right\}$$

is a halfspace in $S^n$, it is convex. $S^n_+$ is the intersection of an infinite number of halfspaces, so it is convex. 

\qed
Intersection: example 2

The set

\[ S = \{ x \in \mathbb{R}^m \mid \sum_{k=1}^{m} x_k \cos kt \leq 1, \forall |t| \leq \pi/3 \} \]

is convex, since it can be expressed as \( S = \bigcap_{|t| \leq \pi/3} S_t \), where \( S_t = \{ x \in \mathbb{R}^m \mid -1 \leq (\cos t, \cdots, \cos mt)^T x \leq 1 \} \).

Figure: The set \( S \) for \( m = 2 \).
**Affine function**

**Theorem**

Suppose \( f : \mathbb{R}^n \to \mathbb{R}^m \) is an affine function (i.e., \( f(x) = Ax + b \)). Then

- the image of a convex set under \( f \) is convex

\[ S \subseteq \mathbb{R}^n \text{ is convex} \implies f(S) = \{ f(x) \mid x \in S \} \text{ is convex} \]

- the inverse image of a convex set under \( f \) is convex

\[ B \subseteq \mathbb{R}^m \text{ is convex} \implies f^{-1}(B) = \{ x \mid f(x) \in B \} \text{ is convex} \]
Affine function: example 1

Show that the ellipsoid

$$\mathcal{E} = \left\{ x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1 \right\}$$

where $P \in S_{++}^n$ is convex.

**Proof.**

Let

$$S = \{ u \in R^n \mid \|u\|_2 \leq 1 \}$$

denote the unit ball in $R^n$. Define an affine function

$$f(u) = P^{1/2}u + x_c$$

$\mathcal{E}$ is the image of $S$ under $f$, so is convex. \qed
Show that the solution set of linear matrix inequality (LMI)

\[ S = \{ x \in \mathbb{R}^n | x_1 A_1 + \cdots + x_n A_n \succeq B \}, \]

where \( B, A_i \in S^m \), is convex.

**Proof.**

Define an affine function \( f : \mathbb{R}^n \to S^m \) given by

\[ f(x) = B - (x_1 A_1 + \cdots + x_n A_n). \]

The solution set \( S \) is the inverse image of the positive semidefinite cone \( S^+_m \), so is convex.
Affine function: example 3

Show that the hyperbolic cone

\[ S = \{ x \in \mathbb{R}^n \mid x^T P x \leq (c^T x)^2, \ c^T x \geq 0 \}, \]

where \( P \in S^n_+ \), is convex.

Proof. Define an affine function \( f : \mathbb{R}^n \to S^{n+1} \) given by

\[ f(x) = (P^{1/2} x, c^T x). \]

The \( S \) is the inverse image of the second-order cone,

\[ \{(z, t) \mid \|z\|_2 \leq t, \ t \geq 0 \}, \]

so is convex. \[ \square \]
Perspective and linear-fractional function

**Perspective function** $P : R^{n+1} \rightarrow R^n$:

$$P(x, t) = \frac{x}{t}, \quad \text{dom } P = \{(x, t) \mid t > 0\}$$

Images and inverse images of convex sets under $P$ are convex.

**Linear-fractional function** $P : R^n \rightarrow R^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}$$

Images and inverse images of convex sets under $f$ are convex.
Generalized inequalities: proper cone

Definition
A cone $K \subseteq \mathbb{R}^n$ is called a **proper cone** if

- $K$ is convex
- $K$ is closed
- $K$ is solid, which means it has nonempty interior
- $K$ is pointed, which means that it contains no line (i.e., $x \in K, -x \in K \implies x = 0$)
Generalized inequalities: proper cone

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Examples:

- nonnegative orthant $K = \mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0, \forall i\}$
- positive semidefinite cone $K = S_+^n$; how about $S_{++}^n$?
- nonnegative polynomials on $[0, 1]$:

  $$K = \{x \in \mathbb{R}^n \mid x_1 + x_2 t + \cdots + x_n t^{n-1} \geq 0, \forall t \in [0, 1]\}$$
A proper cone $K$ can be used to define a **generalized inequality**, a partial ordering on $\mathbb{R}^n$,

$$ x \preceq_K y \iff y - x \in K \quad x \prec_K y \iff y - x \in \text{int} \, K $$

where the latter is called a strict generalized inequality.
Generalized inequalities: definition

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where the latter is called a strict generalized inequality.

Examples:

- componentwise inequality ($K = \mathbb{R}^n_+$)
  $$x \preceq_{\mathbb{R}^n_+} y \iff x_i \leq y_k, \quad \forall i = 1, \ldots, n$$

- matrix inequality ($K = S^n_+$)
  $$x \preceq_{S^n_+} y \iff Y - X \text{ is positive semidefinite}$$
Generalized inequalities: properties

Many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$:

- **transitive**: $x \preceq_K y$, $y \preceq_K z \implies x \preceq_K z$
- **reflexive**: $x \preceq_K x$
- **antisymmetric**: $x \preceq_K y$, $y \preceq_K x \implies x = y$
- **preserved under addition**: $x \preceq_K y$, $u \preceq_K v \implies x + u \preceq_K y + v$
- **preserved under nonnegative scaling**: $x \preceq_K y$, $\alpha \geq 0 \implies \alpha x \preceq_K \alpha y$
- **preserved under limits**: suppose $\lim x_i = x$, $\lim y_i = y$. Then $x_i \preceq_K y_i$, $\forall i \implies x \preceq_K y$
Minimum and minimal elements

$\preceq_K$ is not in general a linear ordering: we can have $x \not\preceq_K y \not\preceq_K x$

$x \in S$ is called **the minimum element** of $S$ with respect to $\preceq_K$ if

$$y \in S \implies x \preceq_K y$$

$x \in S$ is called **the minimal element** of $S$ with respect to $\preceq_K$ if

$$y \in S, \; y \preceq_K x \implies y = x$$
Minimum and minimal elements

$\preceq_K$ is not in general a linear ordering: we can have $x \npreceq_K y \npreceq_K x$

$x \in S$ is called the minimum element of $S$ with respect to $\preceq_K$ if

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$x \in S$ is called the minimal element of $S$ with respect to $\preceq_K$ if

$y \in S, \ y \preceq_K x \implies y = x$

Example:

Figure: $K = R^2_+$. $x_1$ is the minimum element of $S_1$. $x_2$ is the minimal element of $S_2$. 
Separating hyperplane theorem

Theorem

Suppose $C$ and $D$ are two convex sets that do not intersect, i.e., $C \cap D = \emptyset$. Then there exist $a \neq 0$ and $b$ such that

$$a^T x \leq b \text{ for } x \in C, \quad \text{and } a^x b \geq b \text{ for } x \in D$$

The hyperplane $\{x \mid a^x = b\}$ is called a separating hyperplane for $C$ and $D$.

Figure: Examples of convex and nonconvex sets
Supporting hyperplane theorem

Supporting hyperplane to set $C$ at boundary point $x_0$

$$\{x \mid a^x = a^T x_0\}$$

where $a \neq 0$ and satisfies $a^T x \leq a^T x_0$ for all $x \in C$.

Theorem (supporting hyperplane theorem)
If $C$ is convex, then there exists a supporting hyperplane at every boundary point of $C$.

Figure: Examples of convex and nonconvex sets
Definition (dual cones)
Let \( K \) be a cone. The set

\[
K^* = \{ y \mid x^T y \geq 0 \forall x \in K \}
\]

is called the dual cone of \( K \).

Property:
- \( K^* \) is always convex, even when the original cone \( K \) is not (why? intersection of convex sets)
- \( y \in K^* \) if and only if \(-y\) is the normal of a hyperplane that supports \( K \) at the origin
Dual cones: examples

Examples:

- \( K = \mathbb{R}_+^n: \ K^* = \mathbb{R}_+^n \)
- \( K = S^n_+: \ K^* = S^n_+ \)
- \( K = \{(x, t) | \|x\|_2 \leq t\}: \ K^* = \{(x, t) | \|x\|_2 \leq t\} \)
- \( K = \{(x, t) | \|x\| \leq t\}: \ K^* = \{(x, t) | \|x\|_* \leq t\} \)

the first three examples are self-dual cones
Dual of positive semidefinite cone

Theorem

The positive semidefinite cone \( S^+_n \) is self-dual, i.e., given \( Y \in S^n \),

\[
\text{tr}(XY) \geq 0 \quad \forall X \in S^+_n \iff Y \in S^+_n
\]
Theorem

The positive semidefinite cone $S^n_+$ is self-dual, i.e., given $Y \in S^n,$

$$\text{tr}(XY) \geq 0 \quad \forall X \in S^n_+ \iff Y \in S^n_+$$

Proof.

To prove $\iff,$ suppose $Y \notin S^n_+.$ Then $\exists q$ with $q^T Y q = \text{tr}(qq^T Y) < 0,$ which contradicts the lefthand condition.
Dual of positive semidefinite cone

Theorem
The positive semidefinite cone $S_+^n$ is self-dual, i.e., given $Y \in S^n$,

$$\text{tr}(XY) \geq 0 \ \forall X \in S_+^n \iff Y \in S_+^n$$

Proof.
To prove $\implies$, suppose $Y \notin S_+^n$. Then $\exists q$ with

$$q^T Yq = \text{tr}(qq^T Y) < 0,$$

which contradicts the lefthand condition.

To prove $\impliedby$, since $X \succeq 0$, write $X = \sum_{i=1}^n \lambda_i q_i q_i^T$, where $\lambda_i \geq 0$ for all $i$. Then

$$\text{tr}(XY) = \text{tr}(Y \sum_{i=1}^n \lambda_i q_i q_i^T) = \sum_{i=1}^n \lambda_i q_i^T Y q_i \geq 0,$$

because $Y \succeq 0$. \qed
Dual of a norm cone

Theorem

The dual of the cone $K = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\}$ associated with a norm $\| \cdot \|$ in $\mathbb{R}^n$ is the cone defined by the dual norm,

$$K^* = \{(u, s) \in \mathbb{R}^{n+1} \mid \|u\|_* \leq s\},$$

where the dual norm is given by $\|u\|_* = \sup\{u^T x \mid \|x\| \leq 1\}$. 
Theorem

The dual of the cone \( K = \{(x, t) \in \mathbb{R}^{n+1} \mid \|x\| \leq t\} \) associated with a norm \( \| \cdot \| \) in \( \mathbb{R}^n \) is the cone defined by the dual norm,

\[
K^* = \{(u, s) \in \mathbb{R}^{n+1} \mid \|u\|_* \leq s\},
\]
where the dual norm is given by \( \|u\|_* = \sup\{u^T x \mid \|x\| \leq 1\} \).

Proof.
We need to show

\[
x^T u + ts \geq 0 \quad \forall \|x\| \leq t \iff \|u\|_* \leq s
\]

The \( \iff \) direction follows from the definition of the dual norm.

To prove \( \implies \), suppose \( \|u\|_* > s \). Then by the definition of dual norm, \( \exists x \) with \( \|x\| \leq 1 \) and \( x^T u \geq s \). Taking \( t = 1 \), we have \( u^T (-x) + v < 0 \), which is a contradiction.
Dual cones and generalized inequalities

Properties of dual cones: let $K^*$ be the dual of a convex cone $K$.

$\blacktriangleright$ $K^*$ is a convex cone (intersection of a set of homogeneous halfspaces)

$\blacktriangleright$ $K_1 \subseteq K_2 \implies K_2^* \subseteq K_1^*$

$\blacktriangleright$ $K^*$ is closed (intersection of a set of closed sets)

$\blacktriangleright$ $K^{**}$ is the closure of $K$ (if $K$ is closed, then $K^{**} = K$)

$\blacktriangleright$ dual cones of proper cones are proper, hence define generalized inequalities:

$$ y \succeq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succeq_K 0 $$