Optimization problems

- Optimization problems in standard form
- Convex problems in standard form
- Some special problems

Optimization problems in standard form

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \le 0, \quad i = 1, \cdots, m \\ & h_i(\mathbf{x}) = 0, \quad i = 1, \cdots, p \end{aligned}$$

- $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$: optimization variables
- $f_0 : \mathbb{R}^n \to \mathbb{R}$: objective (or cost) function
- $f_i : \mathbb{R}^n \to \mathbb{R}$: inequality constraint functions
- $h_i : \mathbb{R}^n \to \mathbb{R}$: equality constraint functions

Optimal value:

$$p^* = \inf \{ f_0(x) \mid f_i(x) \le 0, i = 1, \cdots, m, h_i(x) = 0, i = 1, \cdots, p \}$$

- $p^* = \infty$ if problem is infeasible
- $p^* = -\infty$ if problem is unbounded below

Implicit contraints

The implicit constraint

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \cap \bigcap_{i=1}^{p} \operatorname{dom} h_i$$

- $\mathcal{D}$ is called the domain of the problem
- the constraints $f_i(x) \leq 0, h_i(x) = 0$ are called explicit constraints
- a problem is unconstrained if it has no explicit constraints

Feasibility problems

$$
\begin{aligned}
\text{find} \quad & f_0(\mathbf{x}) \\
\text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \cdots, m \\
& h_i(\mathbf{x}) = 0, \quad i = 1, \cdots, p
\end{aligned}
$$

can be considered as a special case of the general problem with $f_0(x) = 0$:

$$
\begin{aligned}
\text{minimize} \quad & 0 \\
\text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \cdots, m \\
& h_i(\mathbf{x}) = 0, \quad i = 1, \cdots, p
\end{aligned}
$$

- $p^* = 0$ if the constraints are feasible; any feasible x is optimal
- $p^* = \infty$ if the constraints are infeasible

Convex optimization problems

Convex optimization problems in standard form:

$$\begin{aligned} \text{minimize} \quad & f_0(\mathbf{x}) \\ \text{subject to} \quad & f_i(\mathbf{x}) \leq 0, \quad i = 1, \cdots, m \\ & a_i^T(\mathbf{z}) = b_i, \quad i = 1, \cdots, p \end{aligned}$$

- $f_0, \cdots, f_m$ are convex; equality constraints are affine
- equality constraints are often combined, written as $Ax = b$
- **the feasible set is always convex**

Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron

Linear-fractional program

$$\begin{aligned} \text{minimize} \quad & f_0(x) = \frac{c^T x + d}{e^T x + f} \qquad \text{dom } f_0 = \{x \mid e^T x + f > 0\} \\ \text{subject to} \quad & Gx \preceq h \\ & Ax = b \end{aligned}$$

Define variables $y = \frac{x}{e^T x + f}$ and $z = \frac{1}{e^T x + f}$. Then the problem can be transformed to a LP (variable y, z)

$$\begin{aligned} \text{minimize} \quad & c^T y + dz \\ \text{subject to} \quad & Gh \preceq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0 \end{aligned}$$

Quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in S^n_+$; the objective function is convex quadratic
- feasible set is a polyhedron

Quadratically constrained quadratic program (QCQP)

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{2}x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} \quad & \frac{1}{2}x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \cdots, m \\
& Ax = b
\end{aligned}
$$

- $P_i \in S_+^n$; the objective and the constraints are convex quadratic
- if $P_1, \cdots, P_m \in S_{++}^n$, feasible regions is intersection of m ellipsoids and an affine set

Second-order cone programming (SOCP)

$$\begin{aligned} \text{minimize} \quad & f^T x \\ \text{subject to} \quad & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \cdots, m \\ & Fx = g \end{aligned}$$

where $A_i \in R^{n_i \times n}$, $F \in R^{p \times n}$.

- inequalities are called second-order cone (SOC) constraints:

$$(A_i x + b_i, c_i^T x + d_i) \in \text{ second-order cone in } R^{n_i+1}$$

- reduces to LP if $n_i = 0$; reduces to QCQP if $c_i = 0$

Geometric programming

monomial function

$$f(x) = c x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad \mathrm{dom}\, f = R_{++}^n$$

where $c > 0$, $\alpha_i \in R$

posynomial function: sum of monomials

$$f(x) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}}, \quad \mathrm{dom}\, f = R_{++}^n$$

geometric program (GP)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \quad i = 1, \cdots, m \\ & h_i(x) = 1, \quad i = 1, \cdots, p \end{array}$$

with f_i posynomial, h_i monomial, where $x \succ 0$

Geometric programming in convex form

change variables to $y_i = \log x_i$, and take logarithm of objective, constraints

monomial function $f(x) = c x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ transforms to

$$\log f(e^{y_1}, \cdots, e^{y_n}) = a^T y + b \quad (b = \log c)$$

posynomial function $f(x) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} \cdots x_n^{\alpha_{nk}}$ transforms to

$$\log f(e^{y_1}, \cdots, e^{y_n}) = \log \left(\sum_{k=1}^{K} \exp(a_i^T y + b_i) \right) \quad (b_i = \log c_k)$$

geometric program (GP) transforms to convex problem

$$\begin{array}{ll} \text{minimize} & \log \left(\sum_{k=1}^{K} \exp(a_{0k}^T y + b_{0k}) \right) \\ \text{subject to} & \log \left(\sum_{k=1}^{K} \exp(a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \cdots, m \\ & Gy + d = 0 \end{array}$$

Convex problem with generalized inequality constraints

$$
\begin{aligned}
\text{minimize} \quad & f_0(\mathbf{x}) \\
\text{subject to} \quad & f_i(\mathbf{x}) \preceq_{K_i} 0, \quad i = 1, \cdots, m \\
& Ax = b
\end{aligned}
$$

where $f_0 : R^n \to R$ convex; $f_i : R^n \to R^{k_i}$ K_i-convex wrt proper cone K_i.

- the feasible set is convex
- any locally optimal point is also globally optimal
- the optimality condition for differential f_0 stays the same

Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \qquad Ax = b \end{array}$$

with $G, F_1, \cdots, F_n \in S^k$, and $A \in R^{p \times n}$.

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints (by forming block diagonal matrices)

LP as SDP

$$\text{LP}: \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

written as a SDP:

$$\text{SDP}: \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \operatorname{diag}(Ax - b) \preceq 0 \end{array}$$

SOCP as SDP

$$\text{SOCP}: \quad \begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T X + d_i, \quad i = 1, \cdots, m \end{array}$$

written as a SDP:

$$\text{SDP}: \quad \begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0, \quad i = 1, \cdots, m \end{array}$$