Topics

With a focus on **statistical methods, data mining, and machine learning**

Thursday: (covered by me)
- matlab tutorial
- intro to probability and statistics
- data analysis: clustering (k-means, hierarchical, EM) and PCA

Friday: (covered by Prof Alex Ihler)
- intro to machine learning
- regression methods
- classification methods: k-nearest neighbor, naive Bayes, decision tree, perceptron
- graphical models: hidden Markov models, Gaussian graphical models, Bayesian network
MATLAB tutorial
Exercise 1: logistic map

The logistic map is a polynomial mapping used to model population size in discrete time. It is mathematically written as

\[ x_{n+1} = rx_n(1 - x_n) \]

where

- \( x_n \in [0, 1] \) represents the population at year \( n \) (\( x_0 \) represents the initial population, at year 0)
- \( r > 0 \) represents a combined rate of reproduction and starvation

Exercise:

- Write a MATLAB function \( f(r, x_0, n) \) to return the population values of the first \( n \) years with an initial value of \( x_0 \) and a fixed parameter \( r \)
- Plot and observe the dynamics of the population size for different choices of \( r \):
  - \( r = 0.5 \)
  - \( r = 1.5 \)
  - \( r = 2.5 \)
  - \( r = 3.2 \)
  - \( r = 4 \)
Write a MATLAB function to implement:

\[ f(x, k) = \sqrt{\sqrt{\cdots \sqrt{x}}} \]

where \( k \) is the number nested square roots.
Multistability

Consider the lysis-lysogeny switch model in phage lambda

\[
\frac{dx}{dt} = \frac{\alpha x^2}{1 + (1 + \sigma_1)x^2 + \sigma_2 x^4} - \gamma x + 1
\]

Exercise: choose \( \alpha = 50, \gamma = 20, \sigma_1 = 1, \sigma_2 = 5 \)

- Write a matlab code to display the trajectory of \( x \)
- Try different initial conditions and observe the steady states of the system
Consider the following two coupled differential equations:

\[
\begin{align*}
\dot{x} &= -x + ay + x^2 Y \\
\dot{y} &= b - ay - x^2 y
\end{align*}
\]

Exercise:

- Write down the fixed point of the system
- Determine the stability condition of the fixed point
- Implement a MATLAB code to display the trajectory of the states. Compare the behaviors of the system for different choices of \(a\) and \(b\). In particular, can you find a stable limit cycle solution?
Consider the following two coupled differential equations:

\[ \dot{x} = -x + ay + x^2 y \]
\[ \dot{y} = b - ay - x^2 y \]

The nullclines are

\[ y = \frac{x}{a + x^2} \]
\[ y = \frac{b}{a + x^2} \]

There is only one fixed point \((x^*, y^*) = (b, \frac{b}{a+b^2})\). The condition for the fixed point to be stable is:

\[ \Delta = a + b^2 > 0 \]
\[ \tau = -\frac{b^4 + (2a - 1)b^2 + (a + a^2)}{a + b^2} > 0 \]

When these conditions are not satisfied, the system exhibits stable oscillations.
Reaction-diffusion equations

Consider the following reaction-diffusion system, often called the 'Turing-Gierer-Meinhardt theory',

\[
\begin{align*}
\frac{\partial a}{\partial t} &= r_a + k_a \frac{a^2}{i} - \gamma_a a + D_a \frac{\partial^2 a}{\partial x^2} \\
\frac{\partial i}{\partial t} &= k_i a^2 - \gamma_i i + D_i \frac{\partial^2 i}{\partial x^2}
\end{align*}
\]

where

- $a$ and $i$ are the concentrations of an activator and an inhibitor
- the activator is produced at a constant rate $r_a$
- the activator operates as a dimer while the inhibitor operates as a monomer
- $\gamma_a$ and $\gamma_i$ are first-order decay parameters; $D_a$ and $D_i$ are diffusion constants
Reaction-diffusion equations: simplified

Choosing dimensionless variables yields

\[ \frac{\partial A}{\partial \tau} = 1 + R \frac{A^2}{I} - A + \frac{\partial^2 A}{\partial s^2} \]
\[ \frac{\partial I}{\partial \tau} = Q(A^2 - I) + P \frac{\partial^2 I}{\partial s^2} \]

Exercise:

- Write down the spatially homogenous solution of the system
- Determine the stability condition of the spatially homogenous solution
- Implement a MATLAB code to display the trajectory of the states. Compare the behaviors of the system for different choices of \( R, P, Q \).
Reaction-diffusion equations: analysis

Spatially homogeneous solutions:

\[ \bar{A} = R + 1 \quad \bar{I} = (R + 1)^2 \]

The stability matrix evaluated at the homogeneous solution is

\[
\begin{bmatrix}
\frac{2R\bar{A}}{I} & -\frac{R\bar{A}^2}{I^2} \\
2\bar{A}Q & -Q
\end{bmatrix}
= \begin{bmatrix}
\frac{R-1}{R+1} & -\frac{R}{(R+1)^2} \\
2(R + 1)Q & -Q
\end{bmatrix}.
\]

Thus the fixed point solution is stable only if

\[
\frac{R - 1}{R + 1} < Q \quad Q > 0
\]
Spatially inhomogeneous solutions: Explore the stability of the system around the homogenous solutions. Let \( A(s, \tau) = \bar{A} + A'(s, \tau), \ I(s, \tau) = \bar{I} + I'(s, \tau) \).

We have

\[
\frac{\partial A'}{\partial \tau} = \frac{R - 1}{R + 1} A' - \frac{R}{(R + 1)^2} I' + \frac{\partial^2 A'}{\partial s^2}
\]

\[
\frac{\partial I'}{\partial \tau} = 2Q(1 + R)A' - QI' + P \frac{\partial^2 I}{\partial s^2}
\]

Try the solution in the form of \( A'(s, \tau) = \hat{A}(\tau) \cos(s/l), \ I'(s, \tau) = \hat{I}(\tau) \cos(s/l) \). We have

\[
\frac{d \hat{A}}{\partial \tau} = \left( \frac{R - 1}{R + 1} - \frac{1}{l^2} \right) \hat{A} - \frac{R}{(R + 1)^2} \hat{I}
\]

\[
\frac{d \hat{I}}{\partial \tau} = 2Q(1 + R)\hat{A} - (Q + \frac{P}{l^2})I'
\]

This yields the stability condition:

\[
\frac{Q}{P} l^4 + \left( \frac{Q}{P} - \frac{R - 1}{R + 1} \right) l^2 + 1 > 0
\]

which is satisfied if \( \frac{Q}{P} > \frac{R-1}{R+1} \).