READING: [FW] Chapters 1: Sections 1 - 4


1. On a small island, 250 people have been exposed to an infectious disease. Consider an experiment which consists of determining the number $N$ of people on the island who have the disease.

   a) What is the sample space of the experiment?
   b) Will this space be an equiprobable space?
   c) If $E$ is the event that $N \leq 50$, what is $E^c$ the complement of $E$?
   d) If $A$ us the event $N \geq 40$, what are $A^c$, $A \cup E$ and $A \cap E$?

2. Show $P(E_1 \cup E_2) \leq P(E_1) + P(E_2)$ for any two events $E_k$, not necessarily elementary.

3. Show that the eigenvalues of $M^T$ is the same as that of $M$ and the eigenvector of the transition matrix $M^T$ is the transpose of the eigenvector of $M$.

4. If $x(0) = p$ is a probability vector, prove that $x(n) = M^np$ is also.

5. Prove that the product of two probability matrices is a probability matrix. In particular, any power of a probability matrix is a probability matrix.

6. If $M > O$, show $y = Mx > 0$ for any probability vector $x$. ($A$ matrix $M > O$ means that all elements of $M$ are positive. A vector $p > 0$ means all components of the vector are positive.)

7. If $M$ is the transition matrix of a regular Markov chain and $x(0) = p$ is a probability vector, show that $x(n) = M^n p$ is a positive probability vector for sufficiently large $n$.

8. Obtain $x(n)$ of the IVP for the first order linear system of difference equations $x(n+1) = Mx(n)$, with $x(0) = p = (p_1, 1 - p_1)^T$ and

   $$M = \begin{bmatrix} p_d & p_q \\ 1 - p_d & 1 - p_q \end{bmatrix}.$$ 

9. Determine the eigen-pairs of the transition matrix for the social mobility problem,

   $$M = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.3 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix},$$

   and use it to solve the initial value problem with $x(0) = c = (c_1, c_2, c_3)^T$.

10. Prove that the Markov chain defined by the transition matrix $A$ below is not a regular Markov Chain:

    $$A = \begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix}.$$