Math 291B/CS295 Assignment II Spring, 2011

READING: [FW] Ch. 1: Sec. 5 – 8; Ch. 2: Sec.1 – 2
EXERCISES: Hand in Problems 1, 2, 4 – 7, 9 and 10 to Professor Xie on Monday, April 11, 2011.

1. In an examination consisting of 100 (true-false) questions, statistics show that if a question is answered correctly, the probability of a correct answer on the next question is ¾. On the other hand, if a question is answered incorrectly, the probability of getting the correct answer on the next question is only ¼ . Formulate a Markov chain model and give an estimate for the average score on this examination.

2. In an epidemic, half of those who are well becomes sick each month and a quarter of those who are sick dies each month. (You may assume the other three quarter remains sick. Alternatively, you let ½ recover and ¼ remain sick.) Let (w_n, s_n, d_n) be the probability distribution vector for the three groups of (well, sick and dead) people in the n^{th} monthly survey and formulate a Markov chain model to determine the limiting distribution for this chain.

3. Today, it is relatively quick to compute M^k with the right computer software such as Mathematica. Still it is more efficient and visual to make use of similarity transformations to get the answer.

a) Find the matrix P so that $M^k = [V\Lambda V^{-1}]^k = V\Lambda^k V$ where Λ is the diagonal matrix with the eigenvalues of M lined up along its diagonal

b) Show that the 4 × 4 transition matrix P for Gambler's ruin is not power-positive (hence the corresponding Markov chain is not regular).

4. Show that the Markov chain with either of the following transition matrices is neither regular nor absorbing:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$$

5. An experiment must be repeated until two successful experiments have been completed. Suppose that the probability of a successful experiment on one trial is 1/3.

a) Formulate the transition matrix for an absorbing Markov chain of three possible states.

b) If the first experiment fails, what is the probability that the first success occurs on the fourth trial?

c) What is the expected number of times that the experiment will be repeated?

6. Suppose in selectively breeding show dogs, the breeder crossbreeds a dog of unknown genotype with a hybrid. One offspring chosen at random is mated with a hybrid and the same process is repeated in succeeding generations. The possible states (genotypes) of a randomly selected offspring are dominant (D_m, D_f), hybrid (D_m, R_f) or (D_f, R_m) and recessive (R_m, R_f).

a) Construct the transition matrix M.

b) Obtain a limiting distribution for the Markov chain $\mathbf{x}(n+1) = M\mathbf{x}(n)$.

c) Without determining the solution of the IVP for the linear difference system (which you will obtain in the next problem), explain why is the equilibrium distribution is expected to be independent of the initial distribution

7. For a prescribed initial distribution $\mathbf{x}(0) = \mathbf{p}$, solve the IVP problem of the Markov chain of Problem 6.

8. Suppose a randomly selected offspring from crossbreeding a dog of unknown genotype with a hybrid is mated with a dog of recessive genotype instead. A randomly selected offspring is again crossbred with a recessive and the same process continues in each generation. After many generations, what is the probability that an offspring chosen at random is recessive?

9. A man walks along a four-block stretch of a street in a small town. If he is at corner 2, 3, or 4, then he walks to the left or right with equal probability. He continues until he reaches corner 5, which is a bar, or corner 1, which is his home. If he reaches either home or the bar, he stays there. Formulate the Markov chain model for this random walk problem and find the absorption matrix.

10. If all rows and columns of an $m \times m$ transition matrix M of a *regular* Markov chain sum to 1, show that for $m = 2$ and $m = 3$ the chain's fixed limiting probability distribution vector is $(½, ½)^T$ and $(1/3, 1/3, 1/3)^T$, respectively.