READING:  [FW] Ch. 7 and 8.  

EXERCISES:  With Problems 4 and 7 worth 20 points each, do enough problems to get 80 points and hand them by Wednesday, May 25, 2011.

1. Find the power spectral density of the following autocorrelation functions:

   \[ C_{XX}(\tau) = e^{-2|\tau|}, \quad b) \quad C_{XX}(\tau) = \begin{cases} 1 - \frac{\tau}{T} & (|\tau| < T) \\ 0 & \text{(otherwise)} \end{cases} \]

2. Suppose \( \varphi \) is a random variable with characteristic function \( \Phi(\omega) = E[e^{i\omega \varphi}] \) and \( X(t) = \cos(t + \varphi) \). Show that \( X(t) \) is stationary in the wide sense if and only if \( \Phi(1) = \Phi(2) = 0 \).

3. (This problem is optional and may be used as a replacement for any of the other problems except Problem 4.) Obtain by the PDE method of Liouville:

   \[
   \frac{\partial p_{x}}{\partial t} + \sum_{k=1}^{n} \frac{\partial (p_{x} f_{k})}{\partial x_{k}} = 0, \quad p_{x}(x,0) = p(x^o),
   \]

   the density function \( p_{x}(x,t) \) of the solution process for the following scalar stochastic IVP:

   \[ \begin{align*}
   a) \quad & X' = aX, \quad X(0) = X^o, \\
   b) \quad & X' = aX^2, \quad X(0) = X^o.
   \end{align*} \]

   (Note that the solution for these two problems have already been obtained in class by the conventional method of Theorem 9 of Chapter 3.)

4. For the linear oscillator governed by the second order linear ODE

   \[ X''(t) + \omega^2 X(t) = 0, \quad X(0) = X^o, \quad X'(0) = V^o \]

   where \( \omega \) is a known constant.

   a) Write the IVP as one for a first order system \( \mathbf{X}' = \mathbf{A} \mathbf{X}, \quad \mathbf{X}(0) = (X^o, V^o)^T \) for a vector process \( \mathbf{X}(t) = (X_1(t), X_2(t))^T \) by setting \( X_1 = X \) and \( X_2 = X' \) and \( \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \). Show that the fundamental matrix solution of the ODE is

   \[
   \Phi(t,0) = \begin{bmatrix} \cos(\omega t) & \omega^{-1} \sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{bmatrix}, \quad \Phi^{-1}(t,0) = \begin{bmatrix} \cos(\omega t) & -\omega^{-1} \sin(\omega t) \\ \omega \sin(\omega t) & \cos(\omega t) \end{bmatrix}.
   \]

   b) For the case where \( X^o \) and \( V^o \) are two random variables with a joint density function \( p(X^o,V^o) \), show that

   \[ p_{X}(x,x';t) = p(x \cos(\omega t) - \omega^{-1} x' \sin(\omega t), \omega x \sin(\omega t) + x' \sin(\omega t)). \]

   c) Suppose \( X^o \) and \( V^o \) are i.i.d. Gaussian random variables with mean zero and variance \( \sigma_{x^o}^2 \) and \( \sigma_{v^o}^2 \), respectively. The joint density of these processes is then
\[ p(x^o,\nu^o) = \frac{1}{2\pi\sigma^o_x\sigma^o_{\nu_0}} e^{-\frac{1}{2}\left(\frac{x^o}{\sigma^o_x}\right)^2 - \frac{1}{2}\left(\frac{\nu^o}{\sigma^o_{\nu_0}}\right)^2}. \]

Show that
\[ p(x,x';t) = \frac{1}{2\pi\sigma^o_x\sigma^o_{x'}} e^{-\frac{1}{2}\left(\frac{x-x'}{\sigma^o_x\sigma^o_{x'}}\right)^2}\]

with
\[ g(x,x';t) = \left(\frac{x'}{\sigma^o_x}\right)^2 + \frac{x^2}{\sigma^o_{x'}} - 2\rho(t)x_x'\sigma^o_x\sigma^o_{x'} \]
\[ \sigma^o_x(t) = \sigma^2_{x0}\cos^2(\omega t) + \sigma^2_{\nu_0}\omega^2\sin^2(\omega t) \]
\[ \sigma^o_{x'}(t) = \sigma^2_{x0}\cos^2(\omega t) + \sigma^2_{\nu_0}\omega^2\sin^2(\omega t) \]
\[ \rho(t) = (\sigma^o_x - \omega^2\sigma^o_{x0})\cos(\omega t)\sin(\omega t)/\omega\sigma_{x0}\sigma_{\nu_0} \]

5. \( X^*(t) + X(t) = f(t), \quad X(0) = X'(0) = 0. \)

a) Rewrite the IVP as a first order system in vector form \( \mathbf{X}' = \Lambda \mathbf{X} + \mathbf{F}(t), \mathbf{X}(0) = (0,0)^T \) for a vector process \( \mathbf{X}(t) = (X_1(t),X_2(t))^T \) by setting \( X_1 = X \) and \( X_2 = X' \). Obtain the matrix \( \Lambda \) and the random vector variable \( \mathbf{F}(t)? \)
b) Denote correlation matrix by \( [C_{ij}(t_1,t_2)] \) where \( C_{ij} = C_{X_i,X_j} = <X(t_1)X^T(t_2)>. \) Show that
\[ \frac{\partial}{\partial t_1} [C_{ij}(t_1,t_2)] = A[C_{ij}(t_1,t_2)] + <\mathbf{F}(t_1)\mathbf{X}^T(t_2)>. \]
c) With \( [F_{ij}(t_1,t_2)] = <\mathbf{F}(t_1)\mathbf{X}^T(t_2)>, \) show that
\[ \frac{\partial}{\partial t_2} [F_{ij}(t_1,t_2)] = [A[F_{ij}(t_1,t_2)] + <\mathbf{F}(t_1)\mathbf{F}^T(t_2)>. \]
d) Express the elements of the matrices \( <\mathbf{F}(t_1)\mathbf{X}^T(t_2)> \) and \( <\mathbf{F}(t_1)\mathbf{F}^T(t_2)> \) in terms the components of \( \mathbf{F} \) and \( \mathbf{X} \).

6. \( \frac{dV}{dt} = AV + VA^T + F(t) \)

b) Show that \( F(t) \) is a known quantity if the stochastic forcing \( f \) is temporarily uncorrelated (i.e., delta correlated).

7. a) The impulse response \( h(t,s) \) of the ODE above is its solution for \( f(t) = \delta(t-s) \). Show that
\[ h(t,s) = h(t-s) = \begin{cases} 0 & (t < s) \\ \sin(t-s) & (t > s) \end{cases}. \]
b) Verify that the solution of the IVP is \( x(t) = \int_0^t h(t-s)f(s)ds \).
c) The long time behavior of \( x(t) \) is taken to be \( x(t) = \int_{-\infty}^t h(t-s)f(s)ds \). Suppose \( f(t) \) is a wide sense stationary stochastic process with power spectral density \( S_f(\omega) \). Show that so the (steady state) \( x(t) \).
d) If \( S_x(\omega) \) is the power spectral density of \( X(t) \). Show that \( S_x(\omega) = S_f(\omega)H(i\omega)^2 \), where \( H(i\omega) \) is the Fourier transform of \( h(t) \).
8. For a scalar ODE forced by a scalar Orstein-Uhlenbeck process \( U(t) \), we have the following stochastic IVP:

\[
X'(t) = AX(t) + U(t), \quad U'(t) = -\alpha U(t) + DW(t), \quad X(0) = U(0) = 0.
\]

With \( Y(t) = (X(t), U(t))^T \), obtain an IVP for the correlation matrix \( C_{YY}(t_1, t_2) = \langle Y(t_1) Y^T(t_2) \rangle \) for \( t_1 > t_2 \). with the initial condition given in terms of the variance of \( Y(t) \).

9. Formulate an IVP for the variance \( V(t) = \langle Y(t) Y^T(t) \rangle \) of \( Y(t) \).