Math 291B/CS295 Assignment VII Spring, 2011

READING: [FW] Ch. 7 and 8.
EXERCISES: With Problems 4 and 7 worth 20 points each, do enough problems to get 80 points and hand them by Wednesday, May 25, 2011.

1. Find the power spectral density of the following autocorrelation functions:

$$a)\quad C_{XX}(\tau) = e^{-2\lambda|\tau|}, \qquad b)\quad C_{XX}(\tau) = \begin{cases} 1 - \dfrac{\tau}{T} & (|\tau| < T) \\ 0 & (\text{otherwise}) \end{cases}$$

2. Suppose φ is a random variable with characteristics function $\Phi(\omega) = E[e^{i\omega\varphi}]$ and $X(t) = \cos(t + \varphi)$. Show that $X(t)$ is stationary in the wide sense if and only if $\Phi(1) = \Phi(2) = 0$.

3. (This problem is optional and may be used as a replacement for any of the other problems except Problem 4.) Obtain by the PDE method of Liouville:

$$\frac{\partial p_x}{\partial t} + \sum_{k=1}^{n} \frac{\partial (p_x f_k)}{\partial x_k} = 0, \qquad p_x(\mathbf{x},0) = p(\mathbf{x}^o),$$

the density function $p_x(\mathbf{x},t)$ of the solution process for the following scalar stochastic IVP:

a) $X' = aX, \quad X(0) = X^o,$ b) $X' = aX^2, \quad X(0) = X^o.$

(Note that the solution for these two problems have already been obtained in class by the conventional method of Theorem 9 of Chapter 3.)

4. For the linear oscillator governed by the second order linear ODE

$$X''(t) + \omega^2 X(t) = 0, \qquad X(0) = X^0, \qquad X'(0) = V^0$$

where ω is a known constant.

a) Write the IVP as one for a first order system $\mathbf{X}' = A\mathbf{X}$, $\mathbf{X}(0) = (X^0, V^0)^T$ for a vector process $\mathbf{X}(t) = (X_1(t), X_2(t))^T$ by setting $X_1 = X$ and $X_2 = X'$ and $A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$. Show that the fundamental matrix solution of the ODE is

$$\Phi(t,0) = \begin{bmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{bmatrix}, \qquad \Phi^{-1}(t,0) = \begin{bmatrix} \cos(\omega t) & -\omega^{-1}\sin(\omega t) \\ \omega\sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

b) For the case where X^0 and V^0 are two random variables with a joint density function $p(X^o, V^o)$, show that

$$p_X(x, x'; t) = p(x\cos(\omega t) - \omega^{-1}x'\sin(\omega t),\ \omega x\sin(\omega t) + x'\sin(\omega t).$$

c) Suppose X^0 and V^0 are i.i.d. Gaussian random variables with mean zero and variance σ_{x0}^2 and σ_{v0}^2, respectively. The joint density of these processes is then

$$p(x^o, v^o) = \frac{1}{2\pi\sigma_{x0}\sigma_{v0}} e^{-\frac{1}{2}\left(\frac{x^o}{\sigma_{x0}}\right)^2} e^{-\frac{1}{2}\left(\frac{v^o}{\sigma_{v0}}\right)^2}.$$

Show that

$$p(x, x'; t) = \frac{1}{2\pi\sigma_x\sigma_v[1-\rho(t)]} e^{-g(x,x';t)/2\sigma_x^2\sigma_v^2[1-\rho^2(t)]}$$

with
$$g(x, x'; t) = (x')^2\sigma_x^2 + x^2\sigma_v^2 - 2\rho(t)xx'\sigma_x\sigma_v$$
$$\sigma_x^2(t) = \sigma_{x0}^2\cos^2(\omega t) + \sigma_{v0}^2\omega^{-2}\sin^2(\omega t)$$
$$\sigma_v^2(t) = \sigma_{v0}^2\cos^2(\omega t) + \sigma_{x0}^2\omega^2\sin^2(\omega t)$$
$$\rho(t) = (\sigma_{v0}^2 - \omega^2\sigma_{x0}^2)\cos(\omega t)\sin(\omega t)/\omega\sigma_{x0}\sigma_{v0}$$

5. $$X''(t) + X(t) = f(t), \qquad X(0) = X'(0) = 0.$$

a) Rewrite the IVP as a first order system in vector form $\mathbf{X}' = A\mathbf{X} + \mathbf{F}(t)$, $\mathbf{X}(0) = (0,0)^T$ for a vector process $\mathbf{X}(t) = (X_1(t), X_2(t))^T$ by setting $X_1 = X$ and $X_2 = X'$. Obtain the matrix A and the random vector variable **F**(t)?

b) Denote correlation matrix by $[C_{ij}(t_1,t_2)]$ where $C_{ij} = C_{Xi,Xj} = <\mathbf{X}(t_1)\,\mathbf{X}^T(t_2)>$. Show that
$$\frac{\partial[C_{ij}(t_1,t_2)]}{\partial t_1} = A[C_{ij}(t_1,t_2)] + <\mathbf{F}(t_1)\mathbf{X}^T(t_2)>.$$

c) With $[F_{ij}((t_1,t_2)] = <\mathbf{F}(t_1)\,\mathbf{X}^T(t_2)>$, show that
$$\frac{\partial[F_{ij}(t_1,t_2)]}{\partial t_2} = A[F_{ij}(t_1,t_2)] + <\mathbf{F}(t_1)\mathbf{F}^T(t_2)>.$$

d) Express the elements of the matrices $<\mathbf{F}(t_1)\,\mathbf{X}^T(t_2)>$ and $<\mathbf{F}(t_1)\,\mathbf{F}^T(t_2)>$ in terms the components of **F** and **X** .

6. a) For the stochastic IVP of Problem 5, obtain the following matrix ODE for the covariance matrix $<\mathbf{X}(t)\,\mathbf{X}^T(t)> = [V_{ij}(t)] = V(t)$ and give the elements of $F(t)$ in terms of the components of **F** and **X.**
$$\frac{dV}{dt} = AV + VA^T + F(t)$$

b) Show that F(t) is a known quantity if the stochastic forcing f is temporarily uncorrelated (i.e., delta correlated).

7. a) The *impulse response h(t,s)* of the ODE above is its solution for $f(t) = \delta(t-s)$. Show that
$$h(t,s) = h(t-s) = \begin{cases} 0 & (t < s) \\ \sin(t-s) & (t > s) \end{cases}.$$

b) Verify that the solution of the IVP is $x(t) = \int_0^t h(t-s)f(s)ds$.

c) The long time behavior of $x(t)$ is taken to be $x(t) = \int_{-\infty}^t h(t-s)f(s)ds$. Suppose $f(t)$ is a wide sense stationary stochastic process with power spectral density $F(\omega)$. Show that so the (steady state) $x(t)$.

d) If $S_X(\omega)$ is the power spectral density of X(t). Show that $S_X(\omega) = S_F(\omega)|H(i\omega)|^2$ where $H(i\omega)$ is the Fourier transform of $h(t)$.

8. For a scalar ODE forced by a scalar Orstein-Uhlenbeck process $U(t)$, we have the following stochastic IVP:

$$X'(t) = AX(t) + U(t), \quad U'(t) = -\alpha U(t) + DW(t), \qquad X(0) = U(0) = 0.$$

With $\mathbf{Y}(t) = (X(t), U(t))^T$, obtain an IVP for the correlation matrix $C_{YY}(t_1, t_2) =< \mathbf{Y}(t_1)\mathbf{Y}^T(t_2) >$ for $t_1 > t_2$ .with the initial condition given in terms of the variance of $\mathbf{Y}(t)$.

9. Formulate an IVP for the variance $V(t)\ (= < \mathbf{Y}(t)\mathbf{Y}^T(t) >)$ of $\mathbf{Y}(t)$.