Homework assignment on estimation theory (CS295/Math291B)

Problem 1

Let x and y be random variables such that the random variable x is exponential, and, conditioned on knowledge of x , y is exponentially distributed with parameter x, i.e.,

$$p(x) = \frac{1}{a} e^{-x/a} u(x)$$
$$p(y|x) = x e^{-xy} u(y)$$

where

$$u(t) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

1. Determine $\hat{x}_{\text{BLS}}(y)$, the Bayesian least square estimate of x based on y, $\lambda_{x|y}(y) = E[(\hat{x}_{\text{BLS}}(y) - x)^2|y]$ and $\lambda_{\text{BLS}} = E[(\hat{x}_{\text{BLS}}(y) - x)^2]$.
2. Determine $\hat{x}_{\text{MAP}}(y)$, the MAP estimate of x based on observation of y. Determine the bias and the error variance for this estimator.

Problem 2

Let x be an unknown nonrandom scalar parameter, and suppose that the N-dimensional random vector y represents some observed data, with mean and covariance given by

$$E[y] = cx + d, \qquad \Lambda_y(x) = \Lambda$$

where c, d, Λ are all known.

1. An estimator $\hat{x}(y)$ is linear if $\hat{x}(y) = a^T y + b$ for some a, b.
   Find the unbiased linear estimator $\hat{x}(y)$ with the minimum variance.
2. What is the variance of your estimator in part 1.

Problem 3

Consider the estimation of a nonrandom but unknown parameter x from an observation of the form $y = x + w$ where w is a zero-mean Laplacian random variable, i.e.,

$$p(w) = \frac{\alpha}{2} e^{-\alpha|w|}$$

for some $\alpha > 0$. Does an unbiased estimate of x exist? Explain. Does an efficient estimate of x exist? If so, determine $\hat{x}_{\text{eff}}(y)$. If not, explain.

Problem 4

Suppose x is an unknown parameter and we have N observations of the form

$$y_k = \begin{cases} x + w_k & x \geq 0 \\ 2x + w_k & x < 0 \end{cases}, \qquad k = 1, 2, \cdots, N$$

where the w_k are independent and identically distributed Gaussian random variables with zero mean and variance σ^2.

1. Determine, as a function of x, of the Cramer-Rao bound on the error variance of unbiased estimates of x
2. Does an efficient estimator for x exist? If so, determine $\hat{x}_{\text{eff}}(y_1, y_2, \cdots, y_N)$. If not, explain.
3. Determine $\hat{x}_{\text{ML}}(y_1, y_2, \cdots, y_N)$, the maximum likelihood estimate for x based on $y_1, y_2, \cdots, y_N$.
4. Is the ML estimator consistent? Explain.