ICS 6N Computational Linear Algebra
Systems of Linear Equations

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Linear equations

- Linear algebra was initially developed to solve systems of linear equations

\[ \begin{align*}
2x - y &= 0 \\
-x + 2y &= 3
\end{align*} \]

- We have two equations with two variables \( x \) and \( y \), and we want to find a solution
- First, we ask, is there a solution?
- If there is the system is called **consistent**
- Second, if it is consistent, how many solutions?
The typical process for solving this is to eliminate one of the variables.

First we multiply the second equation by 2 to get an equivalent system of equations (they have the same solution):

\[
\begin{align*}
2x - y &= 0 \\
2 \times (-x + 2y &= 3) \\
\text{⇓} \\
2x - y &= 0 \\
-2x + 4y &= 6
\end{align*}
\]
Second we replace the second equation by the addition of the two equations

\[ 2x - y = 0 \]
\[ 0x + 3y = 6 \]

The solution is then \( y = 2, \ x = 1 \)

There is a solution and it is unique
Geometric interpretation of unique solution

- The solution is the intersection of the lines given by each equation
- We can verify that we got the same solution and it is a **unique solution**
Another case is when we have two parallel lines.
In this case there is no solution since the lines never intersect.
Geometric interpretation of many solutions

- The third case is when one line overlaps the other
- In this case we have infinitely many solutions
Linear equations with three variables
A linear equation in the variables $x_1, x_2, \cdots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $b$ and the coefficients $a_1, \cdots, a_n$ are real numbers that are usually known in advance.

A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables say, $x_1, \cdots, x_n$. 
A system of linear equations has
1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be *consistent* if it has either one solution or infinitely many solutions.

A system of linear equation is said to be *inconsistent* if it has no solution.
Gaussian Elimination

Eliminate variables through three elementary operations:

1) Scale an equation by a nonzero constant: \( C \times R1 \)
2) Replace one equation by the sum of itself and a multiple of another equation: replace \( R3 \) by \( R3 + C \times R2 \)
3) Interchange two equations
Example

\[
x_1 - 2x_2 + x_3 = 0
\]
\[
2x_2 - 8x_3 = 8
\]
\[
-4x_1 + 5x_2 + 9x_3 = -9
\]
4 \times eq1 + eq3 (We eliminate $x_1$ from equation 3) and we get

\begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    2x_2 - 8x_3 &= 8 \\
    -3x_2 + 13x_3 &= -9
\end{align*}
\[ \frac{1}{2} \times eq2 \text{ (we divide equation 2 by 2)} \text{ and we get} \]

\[
\begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    x_2 - 4x_3 &= 4 \\
    -3x_2 + 13x_3 &= -9
\end{align*}
\]
$3 \times eq2 + eq3$ and we get

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= 3
\end{align*}
\]
From here we know how to solve it using backward substitution

\[ x_1 = 2x_2 - x_3 = 29 \]
\[ x_2 = 4 + 4x_3 = 16 \]
\[ x_3 = 3 \]

You can always confirm it is the correct answer by substituting this in the original equation.

As we can see this system is consistent and has a unique solution.
Now we are going to use matrices since we only have to keep track of the coefficients in the equation. Consider the previous example.

- **Coefficient matrix**

\[
\begin{bmatrix}
1 & -2 & 1 \\
0 & 2 & -8 \\
-4 & 5 & 9
\end{bmatrix}
\]

- **Augmented matrix**

\[
\begin{bmatrix}
1 & -2 & 1 & 0 \\
0 & 2 & -8 & 8 \\
-4 & 5 & 9 & -9
\end{bmatrix}
\]
Elementary row operations

Elementary row operations include the following:

- (Replacement) Replace one row by the sum of itself and a multiple of another row.
- (Interchange) Interchange two rows.
- (Scaling) Multiply all entries in a row by a nonzero constant.

Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.
Elementary row operations

- It is important to note that row operations are reversible.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Two fundamental questions about a linear system are as follows:
  - Is the system consistent; that is, does at least one solution exist?
  - If a solution exists, is it the only one; that is, is the solution unique?
Existence and uniqueness of system of equations

Example: determine if the following system is consistent

\[ \begin{align*}
    x_1 - 2x_2 + x_3 &= 0 \\
    2x_2 - 8x_3 &= 8 \\
    -4x_1 + 5x_2 + 9x_3 &= -9
\end{align*} \]

The augmented matrix is

\[
\begin{bmatrix}
    1 & -2 & 1 & 0 \\
    0 & 2 & -8 & 8 \\
    -4 & 5 & 9 & -9
\end{bmatrix}
\]
Gaussian Elimination

\[ 4 \times R_1 + R_3 \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix} \]

\[ \frac{1}{2} \times R_2 \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix} \]

\[ 3 \times R_2 + R_3 \rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \]

Notice the diagonal of ones with zeros below (called upper triangular matrix). This is the shape we are looking for.
Gaussian Elimination

- The corresponding system of equations would be

\[
\begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
x_2 - 4x_3 &= 4 \\
x_3 &= 3
\end{align*}
\]

- The solution is: \( x_3 = 3, \ x_2 = 16, \ x_1 = 29, \) which is what we got before
Another example

\[
\begin{align*}
    x_2 - 4x_3 &= 8 \\
    2x_1 - 3x_2 + 2x_3 &= 1 \\
    5x_1 - 8x_2 + 7x_3 &= 1
\end{align*}
\]

The augmented matrix is

\[
\begin{bmatrix}
0 & 1 & -4 & 8 \\
2 & -3 & 2 & 1 \\
5 & -8 & 7 & 1
\end{bmatrix}
\]

We want to generate an upper triangular matrix, so we have to exchange rows
Gaussian Elimination

\[
\text{exchange } R_1 \text{ and } R_2 \quad \rightarrow \quad \begin{bmatrix}
2 & -3 & 2 & 1 \\
0 & 1 & -4 & 8 \\
5 & -8 & 7 & 1
\end{bmatrix}
\]

\[
\frac{1}{2} \times R_1 \quad \rightarrow \quad \begin{bmatrix}
1 & -1.5 & 1 & 0.5 \\
0 & 1 & -4 & 8 \\
5 & -8 & 7 & 1
\end{bmatrix}
\]

\[
-5 \times R_1 + R_3 \quad \rightarrow \quad \begin{bmatrix}
1 & -1.5 & 1 & 0.5 \\
0 & 1 & -4 & 4 \\
0 & -0.5 & 2 & -1.5
\end{bmatrix}
\]

\[
\frac{1}{2} \times R_2 + R_3 \quad \rightarrow \quad \begin{bmatrix}
1 & -1.5 & 1 & 0.5 \\
0 & 1 & -4 & 4 \\
0 & 0 & 0 & 2.5
\end{bmatrix}
\]
We can convert this to equations and we see in the last one that there is no solution for this system of equations

\[ x_1 - 1.5x_2 + x_3 = 0.5 \]
\[ x_2 - 4x_3 = 8 \]
\[ 0 = 2.5 \]