ICS 6N Computational Linear Algebra
The Matrix Equation $Ax = b$

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Definition: If $A$ is an $m \times n$ matrix, with columns $a_1, \cdots, a_n$, and if $x$ is in $\mathbb{R}^n$, then the product of $A$ and $x$, denoted by $Ax$, is the linear combination of the columns of $A$ using the corresponding entries in $x$ as weights; that is

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Ax is defined only if the number of columns of $A$ equals the number of entries in $x$. 
Example

Let

\[ A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \]

What is \( Ax \)?
Examples

Let

\[ A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \]

What is \( Ax \)?

Solution:

\[ Ax = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 7 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \]
Consider the following system

\[ x_1 + 2x_2 - x_3 = 4 \]
\[ -5x_2 + 3x_3 = 1 \]

Write it as a **matrix equation**

\[
\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}
\]
Matrix equation $Ax = b$

- If $A$ is an $m \times n$ matrix, with columns $a_1, \cdots, a_n$, and if $b$ is in $R^n$, then the matrix equation
  
  $$Ax = b$$

  has the same solution set as the vector equation

  $$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b,$$

  which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

  $$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

- The equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$. 
Computing $Ax$

Let $A$ be an $m \times n$ matrix:

$$A = \begin{bmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
& \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{bmatrix}$$

Then

$$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$
Let $x$ and $y$ be two vectors in $\mathbb{R}^n$. We define the dot product between two vectors as:

$$ x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n $$
Let \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \), \( y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \)

Define \( y^T = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \) - turning a column vector into a row vector

Then

\[ x \cdot y = y^T x \]
Computing $Ax$

$$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} x \cdot 1^{st} \text{ row} \\ x \cdot 2^{nd} \text{ row} \\ \vdots \\ x \cdot m^{th} \text{ row} \end{bmatrix}$$
Example

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \] calculate \( Ax \)

- First way (Using definition):

\[ Ax = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \]

- Second way:

\[
Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}
\]
The matrix with 1s on the diagonal and 0s elsewhere is called an **identity matrix** and is denoted by $I$.

$$
I_n = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ddots & \\
\vdots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 1
\end{bmatrix}
$$

For example:

$$
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

$$
Ix = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x, \text{ for any } x.
$$
If $A$ is an $m \times n$ matrix, $u$ and $v$ are vectors in $\mathbb{R}^n$, and $c$ is a scalar, then

- $A(u + v) = Au + Av$
- $A(cu) = c(Au)$
A system of linear equation is said to be **homogeneous** if it can be written in the form \( Ax = 0 \), where \( A \) is an \( m \times n \) matrix and 0 is the zero vector in \( \mathbb{R}^m \).

- \( x = 0 \) is always a solution, called the trivial solution.
- \( Ax = 0 \) has a nontrivial solution (nonzero vector) if and only if the equation has at least one free variable.
Determine if the following homogeneous system has a nontrivial solution. Describe the solution set.

\[3x_1 + 5x_2 - 4x_3 = 0\]
\[-3x_1 - 2x_2 + 4x_3 = 0\]
\[6x_1 + x_2 - 8x_3 = 0\]
Example

Determine if the following homogeneous system has a nontrivial solution. Describe the solution set.

\[
\begin{align*}
3x_1 + 5x_2 - 4x_3 &= 0 \\
-3x_1 - 2x_2 + 4x_3 &= 0 \\
6x_1 + x_2 - 8x_3 &= 0
\end{align*}
\]

- Reduce the augmented matrix \([A \ 0]\) to echelon form

\[
\begin{bmatrix}
3 & 5 & -4 & 0 \\
-3 & -2 & 4 & 0 \\
6 & 1 & -8 & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & -4/3 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

- Solution: \(x_1 = \frac{4}{3}x_3, \ x_2 = 0\) with \(x_3\) free.

- Written down in vector form: \(x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}\)
**Definition**: An indexed set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \) is called **linearly independent** if

\[
x_1 v_1 + x_2 v_2 + \ldots + x_p v_p = 0
\]

has only the trivial solution.

Otherwise, the set is called **linearly dependent**.
Determine if \( \{v_1, v_2\} \) is linearly independent.

\[
v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
Determine if \( \{ \mathbf{v}_1, \mathbf{v}_2 \} \) is linearly independent.

\[
\begin{align*}
\mathbf{v}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\mathbf{v}_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}
\]
Given an indexed set of vectors \( \{v_1, v_2, \ldots, v_p\} \) in \( \mathbb{R}^n \),

- If the set contains the zero vector, then the set is linearly dependent.
- If the set has only one vector, it is linearly independent.
- If \( p > n \), then the set is linearly dependent.
- If the set with \( p \geq 2 \) are linearly dependent, then at least one of the vectors is a linear combination of the others.
  - Suppose \( x_j \) is not zero, then \( v_j = -\frac{x_1}{x_j} v_1 - \frac{x_2}{x_j} v_2 - \ldots - \frac{x_p}{x_j} v_p \)
  - \( \{v_1, v_2\} \) is linearly dependent if at least one of the vectors is a multiple of the other.
Let \( \{v_1, v_2, \cdots, v_p\} \) be a set of vectors in \( \mathbb{R}^n \), and \( A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix} \), the following statements are equivalent:

a) The set is linearly dependent.
b) \( Ax = 0 \) has nontrivial solutions.
c) \( A \) has at least one free variable.
d) The number of pivots in \( A \) is less than \( p \).
e) \( \text{rank}(A) < p \). Define \( \text{rank}(A) = \text{number of pivots in } A \).
Let \( \{ v_1, v_2, \cdots, v_n \} \) be a set of vectors in \( \mathbb{R}^n \), and \( A = [v_1 \ v_2 \ \cdots \ v_n] \), the following statements are equivalent:

a) The set is linearly independent.

b) \( Ax = 0 \) has only trivial solutions.

c) \( A \) has no free variables.

d) \( \text{rank}(A) = n \). Such a matrix is called **non-singular**.

e) \( Ax = b \) has exactly one solution.