ICS 6N Computational Linear Algebra
The Inverse of a Matrix

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**Definition:** Let $A$ be an $n \times n$ matrix. $A$ is **invertible** if there exists an $n \times n$ matrix $C$ such that

\[ CA = AC = I_n \]

If $A$ is invertible, we denote $C$ by $A^{-1}$, and called it the **inverse** of $A$. 
For example, the following matrix

\[ A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \]

is invertible since we can find a matrix

\[ C = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \]

such that

\[ AC = CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

So

\[ A^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix} \]
The inverse of 2x2 matrices

A 2x2 matrix

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

is invertible if \( ad - bc \neq 0 \), and in this case its inverse is

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

where \( \text{det}(A) = ad - bc \) is called the determinant of A.
The inverse of 2x2 matrices

We can check this

\[
AA^{-1} = \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I
\]

We also obtain the same result for \( A^{-1}A \)
Solution of $Ax=b$

- **Theorem**  If $A$ is an invertible $n \times n$ matrix, then for each $b$ in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

- Thus an invertible matrix is row equivalent to an identity matrix.
Solution of $Ax=b$

- **Theorem**  If $A$ is an invertible $n \times n$ matrix, then for each $b$ in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

- **Proof**
  - A solution exists because $A(A^{-1}b) = AA^{-1}b = Ib = b$.
  - Uniqueness: If there exists another $u$ with $Au = b$, then $A^{-1}Au = A^{-1}b$. So $u = x$. 
Some Properties

- If $A$ is invertible, then $A^{-1}$ is invertible and
  \[(A^{-1})^{-1} = A\]
- If $A$ is invertible, then $A^T$ is invertible and
  \[(A^T)^{-1} = (A^{-1})^T\]
- If $A$ and $B$ are $n \times n$ invertible matrices, so is $AB$ and
  \[(AB)^{-1} = B^{-1}A^{-1}\]
Theorem

Let $A$ be an $n \times n$ matrix, then the following statements are equivalent:

a) $A$ is invertible

b) $A$ is row equivalent to an identity matrix

c) $A$ has $n$ pivot columns

d) $Ax = 0$ only has a trivial solution

e) The columns of $A$ are linearly independent

f) $A^T$ is invertible

g) There is an $n \times n$ matrix $C$ such that $AC = I$

h) There is an $n \times n$ matrix $C$ such that $CA = I$
An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

Let

\[ E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \]

Compute \( E_1A, E_2A, E_3A. \)
Elementary Matrices

- Replacement
  \[ E_1A = \begin{bmatrix}
a & b & c \\
d & e & f \\
-4a + g & -4b + h & -4ci
\end{bmatrix} \]

- Interchange
  \[ E_2A = \begin{bmatrix}
d & e & f \\
a & b & c \\
g & h & i
\end{bmatrix} \]

- Scaling
  \[ E_3A = \begin{bmatrix}
a & b & c \\
d & e & f \\
5g & 5h & 5i
\end{bmatrix} \]
Elementary Matrices

Elementary matrices are invertible

\[ E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}, \]

\[ E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \]

\[ E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}. \]
If an elementary row operation is performed on an mxn matrix A, the resulting matrix can be written as EA, where the matrix E is created by performing the same row operation on $I_m$.

Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.
Theorem: An $n \times n$ matrix $A$ is invertible if and only if $A$ is row equivalent to $I_n$, and in this case, any sequence of elementary row operations that reduces $A$ to $I_n$ also transforms $I_n$ into $A^{-1}$. 
If $A$ is invertible, then through a sequence of elementary row operations, we can reduce $A$ to an identity matrix

$$E_p \ldots E_2 E_1 A = I$$

Thus

$$A^{-1} = E_p \ldots E_2 E_1 = E_p \ldots E_2 E_1 I_n$$

can be interpreted as applying the same sequence row operations to $I_n$.

$$A = E_1^{-1} E_2^{-1} \ldots E_p^{-1}$$
General method for obtaining the inverse of a matrix

- Create the augmented matrix with the identity matrix on the right side
- Reduce the matrix to RREF. If

$$[A \ I] \iff [I \ A^{-1}]$$

Then $A$ is invertible.
Example

Find the inverse of the following matrix, if it exists

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 3 \\
4 & -3 & 8
\end{bmatrix}
\]
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\[
A = \begin{bmatrix}
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\end{bmatrix}
\]

Solution:

\[
\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 3 & 0 & 1 & 0 \\
4 & -3 & 8 & 0 & 0 & 1
\end{bmatrix} \iff \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\
0 & 1 & 0 & -2 & 4 & -1 \\
0 & 0 & 1 & 3/2 & -2 & 1/2
\end{bmatrix}
\]
Theorem: If $A$ is invertible, then $A^{-1}$ is unique.

Proof:
Suppose we can find two matrices $B$ and $C$ such that $AB = BA = I$ and $AC = CA = I$. Then

\[ B = BI = B(AC) = (BA)C = IC = C \]

which implies that $B = C$. So the inverse must be unique. \qed
Under what conditions is $A$ invertible?

If $A$ is a $2\times2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We want to find a $C$ such that $AC = I$.

Let $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$. So we can see $AC = I$ as two systems of linear equations.

$$Ac_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ac_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To solve this we create an augmented matrix but for two systems at the same time, so we consider the augmented matrix for two systems jointly

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$
Example

Consider

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

Reduce the joint augmented matrix to echelon form:

\[
\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}
\]

We have reached RREF and we don’t have free variables, so our solution is unique. So

\[ c_1 = \begin{bmatrix} -2 \\ \frac{3}{2} \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \]

We can confirm that \( AC = CA = I \), so \( A^{-1} = C \)
Find the inverse of a matrix

For any \( n \times n \) matrix \( A \),

\[
AC = I \iff Ac_1 = e_1, Ac_2 = e_2, \ldots, Ac_n = e_n
\]

where \( e_i \) is a column vector whose \( i \)-th entry is 1 and other entries are 0.

Need to solve \( n \) systems of linear equations with the same coefficient matrix.

Method: reduce the following augmented matrix to echelon form

\[
\begin{bmatrix}
a_1 & \cdots & a_n & e_1 & \cdots & e_n
\end{bmatrix}
\]

If all of \( n \) systems are consistent, then \( A^{-1} = C \).
In which case the system has no solution?

Since $A$ is $n \times n$, there at at most $n$ pivot columns

Case 1: The number of pivots $= n$

- We wouldn’t have zeros in the diagonal of the RREF of $A$, so we would always have exactly one solution. This means it would be invertible

Case 2: The number of pivots $< n$

- We would have zeros in the diagonal of the RREF of $A$, so at least one equation would not have a solution. This means it wouldn’t be invertible