An example of a vector with two entries is

\[ W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \]

where \( w_1 \) and \( w_2 \) are any real numbers.

A matrix with only one column is called a column vector, or simply a vector.

The set of all vectors with 2 entries is denoted by \( R^2 \) (read r-two).

Two vectors are equal if and only if their corresponding entries are equal.

Given two vectors \( u \) and \( v \) in \( R^2 \), their sum is the vector \( u + v \) obtained by adding corresponding entries of \( u \) and \( v \).

Given a vector \( u \) and a real number \( c \), the scalar multiple of \( u \) by \( c \) is the vector \( cu \) obtained by multiplying each entry in \( u \) by \( c \).
Given \( u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( v = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \), find \( 4u \), \((-3)v\) and \(4u-3v\).
Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, we can identify a geometric point \((a, b)\) with the column vector \[
\begin{bmatrix}
a \\
b
\end{bmatrix}
\]
So we may regard \(\mathbb{R}^2\) as the set of all points in the plane.
Let $u$ and $v$ be vectors in $\mathbb{R}^n$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{R}^n, \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$$

$au + cv$ is also a vector in $\mathbb{R}^n$

$$a \times \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + b \times \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a \times u_1 + b \times v_1 \\ a \times u_2 + b \times v_2 \\ \vdots \\ a \times u_n + b \times v_n \end{bmatrix}$$
The vector whose entries are all zero is called the **zero vector** and is denoted by $0$.

For all $u, v, w$ in $R^n$ and all scalars $c, d$:

- $u + v = v + u$
- $(u + v) + w = u + (v + w)$
- $u + 0 = 0 + u = u$
- $u + (-u) = -u + u = 0$
- $c(u + v) = cu + cv$
- $(c+d)u = cu + du$
- $c(du) = (cd)u$
- $1u = u$
Given $v_1, v_2, ..., v_p$ vectors in $\mathbb{R}^n$, and given scalars $c_1, c_2, ..., c_p$, then vector $y$ defined by

$$y = c_1 v_1 + c_2 v_2 + ... + c_p v_p$$

is called a linear combination of $v_1, v_2, ...v_p$ with weights $c_1, c_2, ..., c_p$

The weights in a linear combination can be any real numbers, including zero.
Let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $1v_1 + 2v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a linear combination of $v_1$ and $v_2$

- $0v_1 + 0v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a linear combination of $v_1$ and $v_2$
Let \( a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} \), \( a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \), \( b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \). Determine whether \( b \) can be generated (or written) as a linear combination of \( a_1 \) and \( a_2 \). That is, determine whether weights \( x_1 \) and \( x_2 \) exist such that

\[
x_1 a_1 + x_2 a_2 = b
\]
Given \( a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} \), \( a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \), \( b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \), Can \( b \) be written as a linear combination of \( a_1 \) and \( a_2 \) with weights \( x_1 \) and \( x_2 \), i.e., \( x_1 a_1 + x_2 a_2 = b \)?

**Solution:**

- Write down the augmented matrix of the corresponding linear system.
- Row reduce it to an echelon form:

\[
\begin{bmatrix}
1 & 2 & 7 \\
-2 & 5 & 4 \\
-5 & 6 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 7 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]

- There is a solution since there is no pivot in the last column. (The system is consistent)

- The solution is unique: \( x_1 = 3 \), \( x_2 = 2 \). So \( b = 3a_1 + 2a_2 \)
A vector equation

\[ x_1a_1 + x_2a_2 + \cdots + x_na_n = b \]

has the same solution set as the linear system whose augmented matrix is

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_n & b
\end{bmatrix}
\]

In particular, b can be generated by a linear combination of \( a_1, \cdots, a_n \) if and only if there exists a solution to the linear system corresponding to the above matrix.
**Definition** If $v_1, v_2, ..., v_p$ are vectors in $R^n$, then the set of all linear combinations of $v_1, v_2, ..., v_p$, denoted by

$$\text{Span}\{v_1, v_2, ..., v_p\},$$

is called the subset of $R^n$ spanned (generated) by $v_1, \cdots, v_p$.

- Span $\{v_1, v_2, ..., v_p\}$ is the collection of all vectors that can be written in the form

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1, \cdots, c_p$ scalars.

- The zero vector $0$ is always in the Span.
If \( v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), what is \( \text{Span}\{v\} \)?
Example

If \( \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), what is \( \text{Span}\{\mathbf{v}\} \)?

Solution:

- The collection of all vectors in the form of \( c\mathbf{v} = \begin{bmatrix} c \\ c \end{bmatrix} \), with any scalar \( c \).
- Geometrically, it is represented by the line through points \((1, 1)\) and the origin in a plane.
Example

If $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what is $\text{Span}\{v_1, v_2\}$?
Example

If \( \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \) what is \( \text{Span}\{ \mathbf{v}_1, \mathbf{v}_2 \} \)?

Solution: The entire \( \mathbb{R}^2 \) space.
Let $v$ be a nonzero vector in $\mathbb{R}^3$. Then $\text{Span } v$ is the set of all scalar multiples of $v$, which is the set of points on the line in $\mathbb{R}^3$ through $v$ and 0. See the figure below.
If \( u \) and \( v \) are nonzero vectors in \( \mathbb{R}^3 \), with \( v \) not a multiple of \( u \), then \( \text{Span } u, v \) is the plane in \( \mathbb{R}^3 \) that contains \( u \), \( v \), and \( 0 \).

In particular, \( \text{Span } u, v \) contains the line in \( \mathbb{R}^3 \) through \( u \) and \( 0 \) and the line through \( v \) and \( 0 \). See the figure below.