Math 227C: Introduction to Stochastic Differential Equations

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Lecture #12

1 Convergence of Expectaton

We will start with an **example:** Let X_n be $N(0, \frac{1}{n})$ distributed, then the probability density of X_n is $f_n(x) = \sqrt{\frac{n}{2\pi}} e^{-\frac{nx^2}{2}}$. Moreover, $E(X_n) = 0 \ \forall n$.

Let $f(x) = \infty$ if $x = 0$; 0 otherwise. Then $f_n(x) \to f(x)$ pointwise. However, $1 = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n dx \neq \int_{-\infty}^{\infty} f dx = 0$.

In this lecture, we will explore some conditions for which the opposite conclusion will hold.

1.1 Monotone Convergence Theorem

Let $X_n, n = 1, 2, ...$ be a sequence of r.v.'s converging almost surely (a.s.) to a r.v. X. Assume $0 \leq X_1 \leq X_2 \leq ...$ a.s., then $\int_\Omega X dP = \lim_{n\to\infty} \int_\Omega X_n dP$ or equivalently, $E(X) = \lim_{n\to\infty} E(X_n)$.

Definition: We say $X = Y$ a.s. if $P\{\omega \in \Omega : X(\omega) = Y(\omega)\} = 1$.

It follows that $X_n(\omega) \to X(\omega)$ pointwise a.s. if this holds for all ω but a set of measure 0. Similarly, $0 \leq X_1(\omega) \leq X_2(\omega) \leq ...$ a.s. if this holds for all ω but a set of measure 0.

1.2 Dominated Convergence Theorem

Let $X_n, n = 1, 2, ...$ be a sequence of r.v.'s converging a.s. to a r.v. X. Assume that $\exists$ a r.v. Y such that $|X_n| \leq Y$ a.s. for all n, and $E(Y) < \infty$, then $E(X) = \lim_{n\to\infty} E(X_n)$.

1.3 Fatou's Lemma

If $0 \leq X_n \to X$ a.s., then $E(X) \leq \liminf_{n\to\infty} E(X_n)$.

Remark: The three theorem above are also true for a more general class of functions: *measurable functions.*

2 Filtration

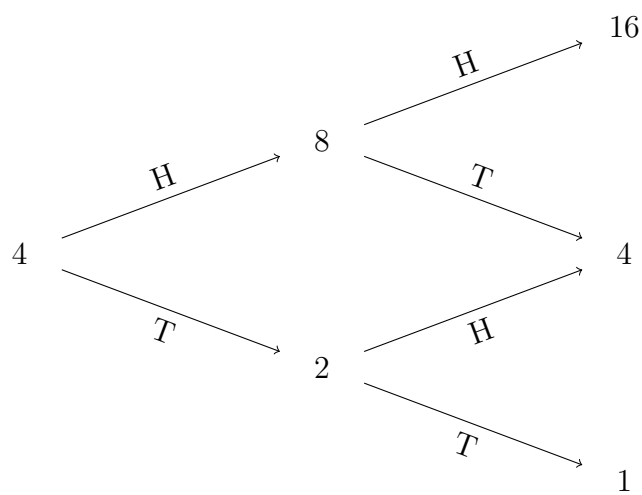
Definition: Let Ω be a nonempty set, and T be a fixed positive number. Further assume that $\forall t \in [0, T]$, there exists a σ-algebra $\mathcal{F}(t)$, and if $s < t$, then $\mathcal{F}(s) \subset \mathcal{F}(t)$, then we can the collection of σ-algebras $\mathcal{F}(t), 0 \leq t \leq T$, a *filtration*.

Definition: Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\mathcal{F}(t), 0 \leq t \leq T$, $X(t)$ be a collection of r.v.'s indexed by $t \in [0, T]$, we say the collection $X(t)$ is an *adapted stochastic process* if for each $t, X(t)$ is $\mathcal{F}(t)$-measurable.

Definition: Let X be a r.v. in $(\Omega, \mathcal{F}, P)$, the σ-algebra generated by X, $\sigma(X)$, is $\{\{X \in B\} : B \in \mathcal{B}\}$. Hence, $\sigma(X)$ is the smallest σ-algebra that X is defined as a r.v..

Definition: Given $(\Omega, \mathcal{F}, P)$ and a r.v. X, we say X is G-measurable if $\sigma(X) \subset G$.

Example: Given the following binomial tree of random r.v.'s $S_n(\omega)$ (S_n can be thought as a stock price),



Then,

- $S_0(\omega) = 4$, and $\sigma(S_0)$=$\mathcal{F}_0$;
- $S_1(\omega) = 8$ if $\omega_1 = H$; 2 otherwise. $\sigma(S_1) = \mathcal{F}_1$;
- $S_2(\omega) = 16$ if $\omega_1 = H, \omega_2 = H$; 4 if $\omega_1 = H, \omega_2 = T$ or if $\omega_1 = T, \omega_2 = H$; 1 otherwise. $\sigma(S_2) = \{\emptyset, \Omega, A_{HH}, A_{HT} \cup A_{TH}, A_{TT}, A_{HH}^C, A_{TT}^C, A_{HH} \cup A_{TT}\} \subsetneq \mathcal{F}_2$.

Note: $\mathcal{F}_i$ are given in the first example of lecture 11 notes.