Math 227C: Introduction to Stochastic Differential Equations

Lecturer: Xiaohui Xie | Lecture #13
Scribe: Alissa Klinzmann | 5/13/14

In this lecture we begin by focusing on a set of definitions.

1 Filtration

Def: A filtration on $(\Omega, \mathcal{F})$ is a family $X = \{X_t\}_{t\geq 0}$ of σ-algebras $X_t \subset \mathcal{F}$ such that $0 \leq s < t \Rightarrow X_s \subset X_t$ (meaning $\{X_t\}$ is increasing).

Given the probability space $(\Omega, \mathcal{F}, P)$, is a filtration process is formed:
• Where the sub-σ-algebra$F_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}$ such that these $\mathcal{F}$'s form a filtration.
• $\{X_t\}_{0\leq t\leq T}$ if X_T is $\mathcal{F}_t$ – measurable (*i.e.*, $\sigma(X_t) \subset \mathcal{F}_t$)

2 Independence

Def: A,B $\in \mathcal{F}$ are independent if P(A $\cap$ B) = P(A)P(B).

Def: Suppose G, H are two sub-σ-algebras of $\mathcal{F}$, we say G,H are independent if P(A $\cap$ B) = P(A)P(B) $\forall$ A$\in$G, B$\in$H.

Def: X,Y (where X,Y are random variables) are independent if σ(X), σ(Y) are independent.

3 Conditional Expectation

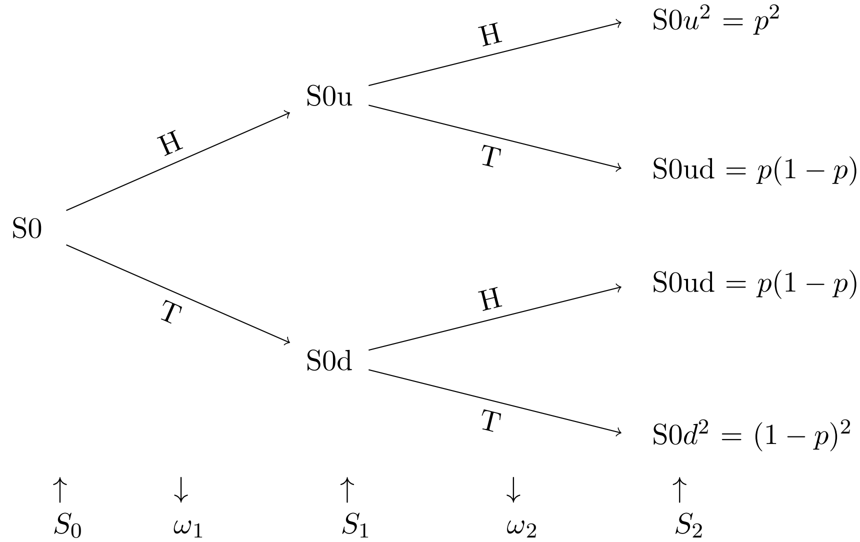
Define E[Y|X] to be a random variable if E[Y|X=X_i] = Y_i.

Def : (Ω,$\mathcal{F}$,P) is a probability space, let G be a sub-σ-algebra (G $\subset \mathcal{F}$), and let X be a random variable that is either non-negative or integrable, then the conditional expectation of X given G, E[X|G], is a random variable that satisfies:

- $E[X|\text{G}]$ is G-measurable (note that E[X|Y] is σ(Y)-measurable which is the estimation of X provided info about Y, meaning E[X|Y] = E[X| σ(Y)]).
- Partial Averaging: $\forall$A$\in$G, $\int_A E[X|\text{G}]\,\text{dp} = \int_A X\, dp$

The existence and uniqueness of E[X| G] comes from the Radon-Nikodym theorem:
Let μ be the measure of G defined by
μ(A) = $\int_A X\, dp$, A $\in$ G.
Then μ is absolutely continuous wrt P|G, so $\exists$ a P|G-unique G-measurable function F on Ω st
μ(A) = $\int_A F\, dp$ $\forall$ A$\in$G.
Thus E[X|G] := F does the job and this function is unique almost surely wrt measure P|G.

Example: Infinite Coin Toss Experiment
Given: $(\Omega_\infty$,F$, P)$ consider the binomial tree (note: S0u = S0-'up', S0d = S0-'down'):



1. G =F$_0$
   $X = S_2 \Rightarrow$ E[S$_2$| $\mathcal{F}_0$] $= E[S_2]$,
   where $E[S_2] =$ [p^2 u^2 + $2p(1-p)$ud + $(1-p)^2$ d^2]S$_0$
   and $E[S_2| \mathcal{F}_0]$ is an estimation of S_2.

2. $E[S_2| \mathcal{F}_1] = \begin{cases} (\text{pu}^2 + (1-p)ud)\text{S}_0 & , \omega_1 = H \\ (\text{pud + (1-p)d}^2)\text{S}_0 & , \omega_1 = T \end{cases}$

3. E[S$_3$| $\mathcal{F}_2$]$\neq$ E[S$_3$| S$_2$] meaning σ(S$_2$)$\subset \mathcal{F}_2$

$$E[S_3| \mathcal{F}_2] = \begin{cases} \vdots & , \omega_1 = H, \omega_2 = H \\ \vdots & , \omega_1 = H, \omega_2 = T \\ \vdots & , \omega_1 = T, \omega_2 = H \\ \vdots & , \omega_1 = T, \omega_2 = T \end{cases}$$

whereas if

$$\text{E[S}_3\text{| S}_2] = \begin{cases} \vdots & , \omega_1 = H, \omega_2 = H \\ \vdots & , \omega_1 = H, \omega_2 = T,\ OR\ \omega_1 = T, \omega_2 = H \\ \vdots & , \omega_1 = T, \omega_2 = T \end{cases}$$

Hence, regarding existence and uniqueness: Yes, there exists a random variable that satisfies the two conditions of Conditional Expectation, and it is usually unique (unless it is at 0).

3.1 Properties of Conditional Expectation

• Let $E[\ E[X|\text{G}]] = \text{E[X]}$ where E[X|G] is a random variable that is an estimation of X (an unbiased estimation).
• If X is G-measurable, then E[X|G] = X.

Suppose X_1 and X_2 are random variables and $\text{a}_1, a_2 \in \mathbb{R}$, then the following properties hold:
1. Linearity: $E[a_1X_1 + a_2X_2|\text{G}] = \text{a}_1E[X_1|\text{G}] + \text{a}_2E[X2|\text{G}]$.
2. Positivity: If X $\geq$ 0 almost surely then E[X|G] $\geq$ 0 almost surely.
3. Taking out what is known: if X is G-measurable, then E[XY|G] = X E[Y|G].
4. Iterated Conditioning: If H $\subset$ G $\subset$ $\mathcal{F}$ then E[E[X|G]|H] = E[X|H].
5. Independence: X is independent of G if E[X|G] = EX.

4 Martingale

Given the probability space (Ω,$\mathcal{F}$,P) and filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$, the stochastic process $\{\text{M}_t\}_{0\leq t\leq T}$ is an adaptive stochastic process with respect to $\{\mathcal{F}_t\}$, then $\{\text{M}_t\}_{0\leq t\leq T}$ is a martingale if $E[M_t|\ \mathcal{F}_s] = M_s$ for 0 $\leq$ s $\leq$ t $\leq$ T, meaning on average the expectation stays the same, neither fluctuating up nor down.
Note:
If $\text{E}[\text{M}_t|\ \mathcal{F}_s] \geq M_s$ then it is sub-martingale.
If $E[M_t|\ \mathcal{F}_s] \leq M_s$ then it is super-martingale.

Example : Looking back to the binomial tree:

$$S_{k+1} = \begin{cases} uS_k & , \omega_k = H, \text{ with probability = p} \\ dS_k & , \omega_k = T, \text{ with probability = 1-p} \end{cases}$$

Is this martingale?

$\text{E}[\text{S}_{k+1}|\ \mathcal{F}_k] = p \cdot uS_k + (1-p) \cdot dS_k = (pu + (1-p)d)S_k$

1.If $pu + (1-p)d = 1,$ then $\{S_k\}$ is martingale.
2. If $pu + (1-p)d \geq 1,$ then $\{S_k\}$ is sub-martingale.
3. If $pu + (1-p)d \leq 1,$ then $\{S_k\}$ is super-martingale.

then it can follow that $E[S_k|\ \mathcal{F}_\ell] = S_\ell,\ \ell < k$.

Brownian Motion :
Thm : $\{B(t)\}_{0\leq t\leq T}$ is a martingale.
Proof : Consider (Ω,$\mathcal{F}$,P)
$\Omega = C[0,T]$ where C = a continuous function
$\mathcal{F}$: generated by $\sigma(B(t))_{0\leq t\leq T}$
Filtration: $\mathcal{F}_t = \sigma(B(t))_{0\leq t\leq T}$ which gives all information of Brownian motion up to time, t.

$E[B(t)|\ \mathcal{F}(\text{s})]\ _{0\leq s\leq t\leq T} = E[B(t) - B(s) + B(s)|\ \mathcal{F}(\text{s})]$
= E[B(t) - B(s)| $\mathcal{F}$(s)] + E[B(s)| $\mathcal{F}$(s)]
= 0 + B(s) = B(s)
since Brownian motion = 0 on average.

Example: Given a random variable, z(t) = $\text{e}^{-\theta B(t)-\frac{1}{2}\theta^2 t}$ where $\theta \in \mathbb{R}$ will be martingale.
To prove this:
Given SDE: $dx(t) = a(x,t)dt + b(x,t)dw$
1. Assume $\mathcal{F}(t)$ is generated by $w(t)$.
2. If $a(x,t) = 0$, then $x(t)$ is a martingale (want the drift term = 0).
WTS $z(t)$ is martingale:

Apply Ito's Formula:
$z(t) = f(t, B(t)) = f(t,x) = e^{-\theta x - \frac{1}{2}\theta^2 t}$

$dz(t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}(dB)^2$

where $\frac{\partial f}{\partial t} = f(-\frac{\theta^2}{2})$, $\frac{\partial f}{\partial x} = f(-\theta)$, $\frac{\partial^2 f}{\partial^2 x} = f(\theta^2)$,

then $dz(T) = f(\frac{-\theta^2}{2})dt + f(-\theta)dB + \frac{1}{2}f(\theta)dt = -\theta f dB$,
showing that the drift terms cancel, therefore is martingale.
$\Rightarrow E[z(t)|\ \mathcal{F}(\text{s})]$ = z(s)
and E[z(t)] = E[z(t)| $\mathcal{F}$(0)] = z(0) = e^0 = 1.