1 Martingale Continued

**Thm:** $z(t) = e^{\theta B(t) - \frac{1}{2} \theta^2 t}$ is a martingale, with $B = \text{Brownian motion}$. 
Where a martingale is $E[M(t)|\mathcal{F}(s)] = M(s) \forall s \leq t \leq T$.

We want to consider the Exit Time: "First Passage Time" or more generally "Stopping Time".  
Fix $x > 0$, let $	au = \min\{t \geq 0 : B(t) = x\}$.
If $M(t)$ is a martingale, then the stopped martingale is still a martingale.

Define:

$$M(t \wedge \tau) = \begin{cases} M(t), & t \leq \tau \\ M(\tau), & t > \tau \end{cases}$$

where $M(t \wedge \tau) = \min(t, \tau)$ and $\tau$ is a random variable.

If we fix $\theta > 0$, then $z(t) = e^{\theta B(t \wedge \tau) - \frac{1}{2} \theta^2 (t \wedge \tau)}$, with $B(t, \tau) \leq x$, is martingale.

which $\Rightarrow E[z(t)] = E[e^{\theta B(t \wedge \tau) - \frac{1}{2} \theta^2 (t \wedge \tau)}] = z(0) = 1$.

Noting that the $\lim_{t \to \infty} e^{-\frac{\theta^2}{2} (t \wedge \tau)} = \begin{cases} 0, & \tau = \infty \\ e^{-\frac{\theta^2}{2} \tau}, & \tau < \infty \end{cases}$
and $z(t)$ is bounded st $0 \leq e^{\theta B(t \wedge \tau) - \frac{\theta^2}{2} (t \wedge \tau)} \leq e^{\theta x}$.

Using dominated convergence theorem, we have $E[\lim_{t \to \infty} e^{\theta B(t \wedge \tau) - \frac{\theta^2}{2} (t \wedge \tau)}] = 1$, with

- $\lim_{t \to \infty} B(t, \tau) = x$ if $\tau < \infty$, and

- $\lim_{t \to \infty} e^{\theta B(t \wedge \tau) - \frac{\theta^2}{2} (t \wedge \tau)} = \begin{cases} 0, & \tau = \infty \\ e^{\theta x - \frac{\theta^2}{2} \tau}, & \tau < \infty \end{cases}$

Hence, $E[e^{\theta x - \frac{\theta^2}{2} \tau} \cdot I_{\{\tau < \infty\}}] = 1$.

1. Let $\theta \to 0$, then $\theta x - \frac{\theta^2}{2} \tau \to 0$ since $\tau < \infty$
$E[I_{\{\tau < \infty\}}] = 1 \Rightarrow P(\tau < \infty) = 1$, since it is almost surely that $\tau < \infty$.

2. $E[e^{-\frac{\theta^2}{2} \tau}] = e^{-\theta x}$,
   let $\alpha = \frac{\theta^2}{2} \Rightarrow E[e^{-\alpha \tau}] = e^{-\sqrt{2\alpha}x}$, $\alpha > 0$.

3. $E[-\tau e^{-\alpha \tau}] = e^{-\sqrt{2\alpha}x}(-\frac{1}{2}(2\alpha)^{-\frac{1}{2}}\sqrt{2}x)$ as $\alpha \to 0$, $E[\tau] = \infty$. 


2 Reflection Principle

This is another method which takes the idea that any time you cross x (shown by the blue line), you can reflect the graph back across the boundary to simplify the distributional properties of Brownian motion.

\[ P(\tau \leq t, B(t) < x) = P(B(t) \geq x) \] (meaning you can flip the graph over the boundary)
\[ \Rightarrow P(\tau \leq t) = P(\tau \leq t, B(t) < x) + P(\tau \leq t, B(t) \geq x) \]
\[ = P(\tau \leq t, B(t) < x) + P(B(t) \geq x) = 2P(B(t) \geq \alpha) \]
\[ = 2 \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}} du \]

with probability density function:
\[ f_\tau(t) = \frac{\partial}{\partial t} P(\tau \leq t) = \frac{x}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}. \]