Math 227C: Introduction to Stochastic Differential Equations

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1 Review

1.1 The Problem

Recall the financial case study involving the European option, there are two ways to derive the Black-Scholes Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\delta^2 X^2 \frac{\partial^2 V}{\partial X^2} + rX\frac{V}{\partial X} - rV = 0 \tag{1}$$

Where $V = V(X, t)$ is the price of the option and $X(t)$ is the amount of the portfolio. This is derived from the underlying stochastic process characterized as:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)dW(t) \tag{2}$$

Where $W(t)$ is a Brownian motion.
We studied two ways to derive (1).

Matching hedging portfolio , where the option of the price is matched exactly to the price of amount of the simple portfolio which consists of stock and money market investment (delta hedging).

Risk neutral derivation , where we use a change of measure to derive the equation in risk-neutral case. This is done by replacing $\mu \to r$ in (2).

1.2 Change of measure

Here we introduce the change of measure:

Definition 1 (Change of Measure)**.** *Let (Ω, F, P) be a probability space and Z be a r.v. in this space such that Z is a.s. non-negative and $E[Z] = 1$, we can redefine measure $\widetilde{P}$ on (Ω, F) such that:*

$$\widetilde{P}(A) = \int_A Z dP$$

Z is also defined as $\frac{d\widetilde{P}}{dP}$, the Radon-Nikodyn derivative of $\widetilde{P}$ w.r.t P.

In addition, if we define $\widetilde{E}[x]$ as the expectation of x using measure $\widetilde{P}$ there is $\widetilde{E}[x] = E[xZ]$ by the above definition.

Definition 2 (Equivalent Measures)**.** *$\widetilde{P}$ is considered equivalent to P, or $\widetilde{P} \equiv P$, if they agree on which sets have measure 0. As a result, they must also agree on sets of measure 1.*

Theorem 1 (Radon-Nikodym Theorem)**.** *Given $\widetilde{P} \equiv P$, there exists a unique $Z = \frac{d\widetilde{P}}{dP}$ as defined above.*

Proof omitted.

Change of measure can be used to make certain r.v. behave in a desired fashion.

Example 1.1. *Let x be a r.v. in (Ω, F, P) and $x \sim N(0,1)$; let $y = x + \theta$ and $Z(x) = e^{\theta x - \frac{1}{2}\theta^2}$. There is now $y \sim N(0,1)$ in the new space $(\Omega, F, \widetilde{P})$ where $Z = \frac{d\widetilde{P}}{dP}$*

Proof. By defn, there is

$$\begin{aligned}\widetilde{f}(y) &= Z(x+\theta)N(x+\theta;0,1)\\ &= e^{\theta x - \theta^2/2}e^{-(x+\theta)^2/2}\frac{1}{\sqrt{2\pi}}\\ &= \frac{1}{\sqrt{2\pi}}e^{\theta^2+\theta x-\theta^2/2-x^2/2-\theta x-\theta^2/2}\\ &= \frac{1}{\sqrt{2\pi}}e^{-x^2/2} = N(x;0,1)\end{aligned}$$

□

Given a Radon-Nikodyn derivative and a filtration we can further define:

Definition 3 (Radon-Nikodyn derivative process)**.** *Let (Ω, F, P) be defined as before, F_t is a filtration of this space up to t with $0 \le t \le T$, T some fixed finish time. Let $\zeta = \frac{d\widetilde{P}}{dP}$ and satisfy the conditions required to be a Radon-Nikodyn derivative. We can define a Radon-Nikodyn derivative process $Z(t)$ as:*

$$Z(t) = E[\zeta|F_t]$$

Note 1. A Radon-Nikodyn derivative process (RNDP) has the following properties:

1. $Z(t)$ is a martingale. This was proven previously.
2. if y is F_t-measurable then there is:

$$\widetilde{E}[y] = E[yZ(t)] = E[yE[\zeta|F_t]] = E[E[y\zeta|F_t]] = E[y\zeta]$$

Remark 1. We can construct a series of Radon-Nikodyn derivative via F_t

3. Let $0 \le s \le t \le T$, y as above, then:

$$\widetilde{E}[y|F_s] = \frac{E[yZ(t)|F_s]}{Z(s)}$$

2 Girsanov Theorem

Given the method of the change of measure and RNDP, we can construct a Brownian motion from an adapted stochastic process.

Theorem 2 (Girsanov Theorem). *Let $W(t)$ be a Brownian motion in (Ω, F, P) and F_t be a filtration with $0 \le t \le T$;*

let $\theta(t)$ be an adapted stochastic process on F_t. Furthermore, define:

$$Z(t) = e^{-\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(u)du, \text{ or}$$

$$d\widetilde{W} = dW + \theta(t)dt$$

Here $Z(t)$ is a r.v. involving the adapted process and we provide two equivalent forms of $\widetilde{W}(t)$ which consists of a Brownian motion term and the adapted process.

Now, denote $Z^t = Z(t)$ and define $\widetilde{P}$ as a change of measure via Z^t.

Then, there is:

$$E[Z^t] = 1 \tag{3}$$

$$\widetilde{W}(t) \text{ is a Brownian motion under } \widetilde{P}. \tag{4}$$

Proof. First, we show that:

$$Z(t) \text{ is a Martingale} \tag{5}$$

Proof. Here we consider $lnZ(t)$:

$$dlnZ^t = -\theta dW - \frac{1}{2}\theta^2 dt$$

,

now let $y(x) = e^x$ and apply Ito's rule on y, there is then:

$$\begin{aligned} dZ^t &= dy(lnZ^t) = Z^t dlnZ^t + \frac{1}{2}Z^t(dlnZ^t)^2 \\ &= Z^t[-\theta dW - \frac{1}{2}\theta^2 dt + \frac{1}{2}\theta^2 dt] \\ &= -Z^t\theta dW \end{aligned}$$

Where we used W being a Brownian motion to cross out a few terms. As a result, since the drift term is completely eliminated from Z^t it is by definition a Martingale. □

Given the above, then it is straightforward to find that $E[Z^t] = Z(0) = 1$.

Second, it is obvious, by virtue of W being a Brownian motion that:

$$(d\widetilde{W})^2 = dW^2 = dt \tag{6}$$

Third, we show that:

$$\widetilde{W} \text{ is a Martingale under } \widetilde{P} \tag{7}$$

Proof. Here we consider $E[\widetilde{W}(t)Z^t|F_s]$ where F_s is a filtration on s such that $0 \leq s \leq t \leq T$. The dynamics of the term $\widetilde{W}(t)Z^t$ is such that (we omit showing that all are functions of t):

$$d(\widetilde{W}Z^t) = \widetilde{W}dZ^t + Z^t d\widetilde{W} + dZ^t d\widetilde{W}$$

By Ito's calculus product rule. Where:

$$\widetilde{W}dZ^t = -\widetilde{W}Z^t\theta dW$$

$$Z^t d\widetilde{W} = Z^t dW + Z^t\theta dt$$

$$dZ^t d\widetilde{W} = -Z^t\theta dW d\widetilde{W} = -Z^t\theta dt$$

Together there is :

$$\begin{aligned}\Rightarrow d(\widetilde{W}Z^t) &= -\widetilde{W}Z^t\theta dW - Z^t\theta dt + Z^t dW + Z^t\theta dt \\ &= Z^t(1 - \widetilde{W}\theta)dW\end{aligned}$$

Once again, since the drift term is eliminated, this dynamics is that of a Martingale and as such we know by definition:

$$E[\widetilde{W}(t)Z^t|F_s] = W(s)$$

Now, given the above, $\widetilde{W}$ is a Martingale.

□

Finally, given (6) and (7), there is (4) by definition; previous (3) was already proven. □

3 Risk neutral change of measure

Now we can derive equation (1) using the risk neutral change of measure. Consider equation (2) along with the following discount due to interest rate:

$$D(t) = e^{-\int_0^t r(s)ds}$$

Then, we examine the dynamics of $D(t)S(t)$, after some manipulation, there is:

$$dDS = \sigma(t)S(t)[dW + \theta(t)dt]$$

, where:

$$\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}$$

Now $\theta(t)$ is an adapted process from which we can derive a $\widetilde{W} = dW + \theta(t)dt$ under the change of method corresponding to $\theta(t)$. By Girsanove's Theorem, we know that $\widetilde{W}$ is a Brownian motion under the new measure, which represents the risk neutral world in which the drift rate of the stock price is already discounted by the interest rate.

Note 2. After the change of measure, the followings are true:

1. $D(t)S(t)$ is a Martingale.
2. $D(t)X(t)$ is also a Martingale.
3. We can set the price of the option, discounted by $D(t)$, as the expectation under the change of measure conditioned on the filtration F_t with finish time (time of option usage) T, or:

$$D(t)V(t) = \widetilde{E}[D(t)V(t)|F_t], 0 \leq t \leq T$$

From the above (1) can be derived.