1 Review

1.1 The Problem

Recall the financial case study involving the European option, there are two ways to derive the Black-Scholes Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 V}{\partial X^2} + rX \frac{V}{\partial X} - rV = 0 \quad (1)$$

Where $V = V(X, t)$ is the price of the option and $X(t)$ is the amount of the portfolio. This is derived from the underlying stochastic process characterized as:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)dW(t) \quad (2)$$

Where $W(t)$ is a Brownian motion.

We studied two ways to derive (1):

- **Matching hedging portfolio**, where the option of the price is matched exactly to the price of amount of the simple portfolio which consists of stock and money market investment (delta hedging).
- **Risk neutral derivation**, where we use a change of measure to derive the equation in risk-neutral case. This is done by replacing $\mu \rightarrow r$ in (2).

1.2 Change of measure

Here we introduce the change of measure:

**Definition 1** (Change of Measure). Let $(\Omega, F, P)$ be a probability space and $Z$ be a r.v. in this space such that $Z$ is a.s. non-negative and $E[Z] = 1$, we can redefine measure $\tilde{P}$ on $(\Omega, F)$ such that:

$$\tilde{P}(A) = \int_A ZdP$$

$Z$ is also defined as $\frac{d\tilde{P}}{dP}$, the Radon-Nikodym derivative of $\tilde{P}$ w.r.t $P$.

In addition, if we define $\tilde{E}[x]$ as the expectation of $x$ using measure $\tilde{P}$ there is $\tilde{E}[x] = E[xZ]$ by the above definition.

**Definition 2** (Equivalent Measures). $\tilde{P}$ is considered equivalent to $P$, or $\tilde{P} \equiv P$, if they agree on which sets have measure 0. As a result, they must also agree on sets of measure 1.

**Theorem 1** (Radon-Nikodym Theorem). Given $\tilde{P} \equiv P$, there exists a unique $Z = \frac{d\tilde{P}}{dP}$ as defined above.
Proof omitted.
Change of measure can be used to make certain r.v. behave in a desired fashion.

**Example 1.1.** Let $x$ be a r.v. in $(\Omega, F, P)$ and $x \sim N(0,1)$; let $y = x + \theta$ and $Z(x) = e^{\theta x - \frac{1}{2} \theta^2}$. There is now $y \sim N(0,1)$ in the new space $(\Omega, F, \tilde{P})$ where $Z = \frac{d\tilde{P}}{dP}$

**Proof.** By defn, there is

$$
\tilde{f}(y) = Z(x + \theta)N(x + \theta; 0, 1) = e^{\theta x - \theta^2/2} e^{-(x+\theta)^2/2} \frac{1}{\sqrt{2\pi}}
$$

$$
= \frac{1}{\sqrt{2\pi}} e^{\theta^2 + \theta x - \theta^2/2 - x^2/2 - \theta x - \theta^2/2}
$$

$$
= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} = N(x; 0, 1)
$$

Given a Radon-Nikodym derivative and a filtration we can further define:

**Definition 3** (Radon-Nikodym derivative process). Let $(\Omega, F, P)$ be defined as before, $F_t$ is a filtration of this space up to $t$ with $0 \leq t \leq T$, $T$ some fixed finish time. Let $\zeta = \frac{d\tilde{P}}{dP}$ and satisfy the conditions required to be a Radon-Nikodym derivative. We can define a Radon-Nikodym derivative process $Z(t)$ as:

$$
Z(t) = E[\zeta | F_t]
$$

**Note 1.** A Radon-Nikodym derivative process (RNDP) has the following properties:

1. $Z(t)$ is a martingale. This was proven previously.

2. if $y$ is $F_t$-measurable then there is:

$$
\tilde{E}[y] = E[yZ(t)] = E[yE[\zeta | F_t]] = E[E[y\zeta | F_t]] = E[y\zeta]
$$

**Remark 1.** We can construct a series of Radon-Nikodym derivative via $F_t$

3. Let $0 \leq s \leq t \leq T$, $y$ as above, then:

$$
\tilde{E}[y | F_s] = \frac{E[yZ(t) | F_s]}{Z(s)}
$$
2 Girsanov Theorem

Given the method of the change of measure and RNDP, we can construct a Brownian motion from an adapted stochastic process.

**Theorem 2** (Girsanov Theorem). Let \( W(t) \) be a Brownian motion in \((\Omega, F, P)\) and \( F_t \) be a filtration with \(0 \leq t \leq T\);

let \( \theta(t) \) be an adapted stochastic process on \( F_t \). Furthermore, define:

\[
Z(t) = e^{-\int_0^t \theta(u) dW(u) - \frac{1}{2} \int_0^t \theta^2(u) du}
\]

\[
\tilde{W}(t) = W(t) + \int_0^t \theta(u) du,
\]

or

\[
d\tilde{W} = dW + \theta(t) dt
\]

Here \( Z(t) \) is a r.v. involving the adapted process and we provide two equivalent forms of \( \tilde{W}(t) \) which consists of a Brownian motion term and the adapted process.

Now, denote \( Z^t = Z(t) \) and define \( \tilde{P} \) as a change of measure via \( Z^t \).

Then, there is:

\[
E[Z^t] = 1
\]

\( \tilde{W}(t) \) is a Brownian motion under \( \tilde{P} \).

**Proof.** First, we show that:

\[
Z(t) \text{ is a Martingale}
\]

**Proof.** Here we consider \( \ln Z(t) \):

\[
d\ln Z^t = -\theta dW - \frac{1}{2} \theta^2 dt
\]

Now let \( y(x) = e^x \) and apply Ito’s rule on \( y \), there is then:

\[
dZ^t = dy(\ln Z^t) = Z^t d\ln Z^t + \frac{1}{2} Z^t (d\ln Z^t)^2
\]

\[
= Z^t [-\theta dW - \frac{1}{2} \theta^2 dt + \frac{1}{2} \theta^2 dt]
\]

\[
= -Z^t \theta dW
\]

Where we used \( W \) being a Brownian motion to cross out a few terms. As a result, since the drift term is completely eliminated from \( Z^t \) it is by definition a Martingale.

Given the above, then it is straightforward to find that \( E[Z^t] = Z(0) = 1 \).

Second, it is obvious, by virtue of \( W \) being a Brownian motion that:

\[
(\tilde{dW})^2 = dW^2 = dt
\]
Third, we show that:

\[ \tilde{W} \text{ is a Martingale under } \tilde{P} \quad (7) \]

Proof. Here we consider \( E[\tilde{W}(t)Z^t|F_s] \) where \( F_s \) is a filtration on \( s \) such that \( 0 \leq s \leq t \leq T \). The dynamics of the term \( \tilde{W}(t)Z^t \) is such that (we omit showing that all are functions of \( t \)):

\[
d(\tilde{W}Z^t) = \tilde{W}dZ^t + Z^t d\tilde{W} + dZ^t d\tilde{W}
\]

By Ito’s calculus product rule. Where:

\[
\tilde{W}dZ^t = -\tilde{W} Z^t \theta dW
\]

\[
Z^t d\tilde{W} = Z^t dW + Z^t \theta dt
\]

\[
dZ^t d\tilde{W} = -Z^t \theta dW d\tilde{W} = -Z^t \theta dt
\]

Together there is:

\[
\Rightarrow d(\tilde{W} Z^t) = -\tilde{W} Z^t \theta dW - Z^t \theta dt + Z^t dW + Z^t \theta dt
\]

\[
= Z^t (1 - \tilde{W} \theta) dW
\]

Once again, since the drift term is eliminated, this dynamics is that of a Martingale and as such we know by definition:

\[
E[\tilde{W}(t)Z^t|F_s] = W(s)
\]

Now, given the above, \( \tilde{W} \) is a Martingale.

\[ \square \]

Finally, given (6) and (7), there is (4) by definition; previous (3) was already proven. \[ \square \]
3 Risk neutral change of measure

Now we can derive equation (1) using the risk neutral change of measure. Consider equation (2) along with the following discount due to interest rate:

\[ D(t) = e^{-\int_0^t r(s)ds} \]

Then, we examine the dynamics of \( D(t)S(t) \), after some manipulation, there is:

\[ dDS = \sigma(t)S(t)[dW + \theta(t)dt] \]

, where:

\[ \theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)} \]

Now \( \theta(t) \) is an adapted process from which we can derive a \( \tilde{W} = dW + \theta(t)dt \) under the change of method corresponding to \( \theta(t) \). By Girsanove’s Theorem, we know that \( \tilde{W} \) is a Brownian motion under the new measure, which represents the risk neutral world in which the drift rate of the stock price is already discounted by the interest rate.

**Note 2.** After the change of measure, the followings are true:

1. \( D(t)S(t) \) is a Martingale.
2. \( D(t)X(t) \) is also a Martingale.
3. We can set the price of the option, discounted by \( D(t) \), as the expectation under the change of measure conditioned on the filtration \( F_t \) with finish time (time of option usage) \( T \), or:

\[ D(t)V(t) = \tilde{E}[D(t)V(t)|F_t], 0 \leq t \leq T \]

From the above (1) can be derived.