Math 227C: Introduction to Stochastic Differential Equations

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1 Continuous Time and Space Stochastic Processes

1.1 Gaussian Distribution

Suppose we have a gaussian distribution such that $x \sim N(\mu, \sigma^2)$ Then we can calculate the probability density function:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

And we have that the expectation of x, $\mathrm{E}[x] = \mu$. and the variance, $\mathrm{var}[(x-\mu)^2] = \sigma^2$. Now, assume that $\mu = 0$. Then we have:

$$\mathrm{E}[x] = 0$$
$$\mathrm{E}[x^2] = \sigma^2$$
$$\mathrm{E}[x^P] = \sigma^2 (p-1)\mathrm{E}[x^{P-2}]$$

where P is an even integer. This last property can be derived using integration by parts.

EXAMPLE:

$\mathrm{E}[x^4] = \sigma^2 \cdot 3\sigma^2 = 3\sigma^4$
$\mathrm{E}[x^P] = \left(\frac{\sigma^2}{2}\right)^{\frac{P}{2}} \frac{P!}{(\frac{P}{2})!}$

1.2 Gaussian Distribution in Partial Differential Equations

Suppose we have the diffusion equation:

$$\frac{\partial p(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(t,x)}{\partial x^2}$$
$$p(0,x) = \psi(x)$$

Then the solution is a Gaussian distribution!

$$p(t,x) = \frac{1}{2\pi t} e^{-\frac{x^2}{2t}} \tag{1}$$

Now, t acts like the variance, with $\sigma^2 = t$. We can find Green's function for (1):

$$p(t,x) = \int \frac{1}{2\pi t} e^{-\frac{(x-y)^2}{2t}} p(0,y)\mathrm{d}y$$

1.3 Gaussian PDE's in $\mathbb{R}^n$

Suppose we are in n dimensional space and matrices $Q = Q^T$ are both positive and finite, and that $Q_{i,j}$ are the entries of Q. Then:

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= \frac{1}{2}\Sigma_{i,j=1}^{n} q_{i,j}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}p(t,x) \\ &= \frac{1}{2}\left(\nabla p(t,x)^T Q \nabla p(t,x)\right)\end{aligned}$$

Then the solution is a multi dimensional gaussian distribution:

$$p(t,x) = \frac{1}{\sqrt{\det(Q)(2\pi t)^n}}\exp(-x^T(2Qt)^{-1}x)$$

Where the convariance matrix, $\Sigma = tQ$ scales linearly with t.

2 Brownian Motion

Robert Brown is credited for describing "Brownian" motion. Brown observed pollen particles moving randomly and described the random motion. Brownian motion is also sometimes called a Wiener process bacause Norbert Wiener was the first person to describe random motion mathematically.

2.1 Bi-directional Poisson Counter

Suppose we have two poisson processes, $dN_1(t)$ and $dN_2(t)$ with rates $\frac{\lambda}{2}$ Then:

$$dy(t) = dN_1(t) - dN_2(t) \tag{2}$$

Now, we rescale (1) so that:

$$\begin{aligned}x_\lambda(t) &= \frac{1}{\sqrt{\lambda}}y(t) \\ dx_\lambda(t) &= \frac{1}{\sqrt{\lambda}}dN_1(t) - \frac{1}{\sqrt{\lambda}}dN_2(t)\end{aligned}$$

where the jump size is proportional to $\frac{1}{\sqrt{\lambda}}$. Then we have

$$\begin{aligned}\frac{dE[x_\lambda(t)]}{dt} &= \frac{\lambda}{2\sqrt{\lambda}}dt - \frac{\lambda}{2\sqrt{\lambda}}dt = 0 \\ &\Rightarrow x_\lambda(0) = 0 \\ &\Rightarrow E[x_\lambda^p(t)] = 0 \text{ (when p is an odd integer, by symmetry)} \\ &\Rightarrow dx_\lambda(t) = \left[(\lambda + \frac{1}{\sqrt{\lambda}})^p - x^p\right]dN_1 + \left[(x - \frac{1}{\sqrt{\lambda}})^p - x^p\right]dN_2\end{aligned}$$

We taylor expand so that:

$$\begin{aligned}&\left[x^p + \binom{p}{1}x^{p-1}\cdot\frac{1}{\sqrt{\lambda}} + \binom{p}{2}x^{p-2}\frac{1}{\lambda} + \cdots - x^p\right]dN_1 \\ &+ \left[x^p + \binom{p}{1}x^{p-1}\cdot\frac{1}{\sqrt{\lambda}} + \binom{p}{2}x^{p-2}\frac{1}{\lambda} + \cdots - x^p\right]dN_2 \\ &= \binom{p}{2}x^{p-2}\frac{1}{\sqrt{\lambda}}(dN_1 + dN_2) \text{ (for expectation)}\end{aligned}$$

Then:

$$\frac{dE[x^p(t)]}{dt} = \frac{p(p-1)}{2} E[x^{p-2}(t)$$

When $\lambda \to \infty$ we can ignore higher order terms in the taylor expansion. Then we have:

1. $\frac{d}{dt} E[x^2(t)] = 1 \Rightarrow E[x^2(t)] = t$ & $dx_\lambda^2(t) = \binom{p}{2}(dN_1 + dN_2)$
2. $\frac{d}{dt} E[x^4(t)] = 2 \cdot 3! \cdot E[x^2(t)] = 6t \Rightarrow E[x^4(t)] = 3t^2$
3. $E[x^p(t)] \frac{p!}{\frac{p}{2}!} \left(\frac{t}{2}\right)^{\frac{p}{2}}$

Let $\sigma^2 = t$ Then all moments match, so as $\lambda \to \infty$ the random variable $x(t)$ will be Gaussian with mean 0 and variance t.