# Math 227C/CS 285 Problem Set 3

Consider the problem of determining the probablity that the solution of

$$dx = f(x)dt + g(x)dw \qquad x(0) \in S_1$$

leaves an open connected set $S \supset S_1$ before time $t$. Such problems arise in the analysis of the life expectancy of a machine, the time to financial ruin, etc, the evolution time to the fixation of mutations in a population, etc. One way to formulate this is to consider a modified process which satisifies the given equation as long as $x \in S$ and satisfies $dx = 0$ once $x$ reaches the boundary of $S$. The corresponding Kolmogorov forward equation in $S$ is

$$\frac{\partial \rho(x,t)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x)\rho(x,t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_j} g_i(x)g_j(x)\rho(x,t)$$

1. What would be an approriate boundary condition for $\rho(x,t)$ on the modified process?
2. Let $T = \inf\{t : x(t) \notin S\}$, which is often called exit time. Write down the probability distribution of exit times in term of $\rho(x,t)$.
3. Now consider an example of one-dimensional process

   $$dx = -xdt + dw \qquad x(0) = 0.$$

   We want to know the probability that $x(t)$ has not left the interval $[-\pi, \pi]$ over the period $[0,t]$. Write down the Kolmogorov forward equation with boundary conditions.
4. Solve the above Kolmogorov equation and find out the probablity distribution of exit times.