

# Monotonic Convergence in an Information-Theoretic Law of Small Numbers

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**Abstract**—An “entropy increasing to the maximum” result analogous to the entropic central limit theorem (Barron 1986; Artstein et al. 2004) is obtained in the discrete setting. This involves the thinning operation and a Poisson limit. Monotonic convergence in relative entropy is established for general discrete distributions, while monotonic increase of Shannon entropy is proved for the special class of ultra-log-concave distributions. Overall we extend the parallel between the information-theoretic central limit theorem and law of small numbers explored by Kontoyiannis et al. (2005) and Harremoës et al. (2007, 2008, 2009). Ingredients in the proofs include convexity, majorization, and stochastic orders.

**Index Terms**—binomial thinning; convex order; logarithmic Sobolev inequality; majorization; maximum entropy; Poisson approximation; relative entropy; Schur-concavity; ultra-log-concavity.

## I. INTRODUCTION

The information-theoretic central limit theorem (CLT, [4]) states that, for a sequence of independent and identically distributed (i.i.d.) random variables  $X_i$ ,  $i = 1, 2, \dots$ , with zero mean and unit variance, the normalized partial sum  $Z_n = \sum_{i=1}^n X_i / \sqrt{n}$  tends to  $N(0, 1)$  as  $n \rightarrow \infty$  in relative entropy, as long as the relative entropy  $D(Z_n | N(0, 1))$  is eventually finite. An interesting feature is that  $D(Z_n | N(0, 1))$  decreases monotonically to zero, or, equivalently, the differential entropy of  $Z_n$  increases to that of the standard normal. While this monotonicity is an old problem ([25]), its full solution is obtained only recently by Artstein et al. [2]; see [35], [27], [32], [33] for ramifications. In this paper we establish analogous monotonicity results for a version of the law of small numbers, extending the parallel between the information theoretic CLT and law of small numbers explored by [24], [15], [16], [17]. Such monotonicity results are interesting as they reveal fundamental connections between probability, information theory, and physics (the analogy with the second law of thermodynamics). Moreover, the associated inequalities are often of great practical significance. The entropic CLT, for example, is closely related to Shannon’s entropy power inequality ([5], [34]), which is a valuable tool in analyzing Gaussian channels.

Informally, the law of small numbers refers to the phenomenon that, for random variables  $X_i$  on  $\mathbf{Z}_+ = \{0, 1, \dots\}$ , the sum  $\sum_{i=1}^n X_i$  has approximately a Poisson distribution with mean  $\lambda = \sum_{i=1}^n EX_i$ , as long as i) each of  $X_i$  is such

that  $\Pr(X_i = 0)$  is close to one,  $\Pr(X_i = 1)$  is uniformly small, and  $\Pr(X_i > 1)$  is negligible compared to  $\Pr(X_i = 1)$ ; and ii) the dependence between the  $X_i$ ’s is sufficiently weak. In the version considered by Harremoës et al. [15], [16], [17] and in this paper, the  $X_i$ ’s are i.i.d. random variables obtained from a common distribution through *thinning*. (Indeed, Harremoës et al. term their result “the law of thin numbers”.) The notion of thinning is introduced by Rényi [30].

**Definition 1:** The  $\alpha$ -thinning ( $\alpha \in (0, 1)$ ) of a probability mass function (pmf)  $f$  on  $\mathbf{Z}_+$ , denoted as  $T_\alpha(f)$ , is the pmf of  $\sum_{i=1}^Y X_i$ , where  $Y$  has pmf  $f$  and, independent of  $Y$ ,  $X_i$ ,  $i = 1, 2, \dots$ , are i.i.d. Bernoulli( $\alpha$ ) random variables, i.e.,  $\Pr(X_i = 1) = 1 - \Pr(X_i = 0) = \alpha$ .

Thinning is closely associated with certain classical distributions such as the Poisson and the binomial. For the Poisson pmf  $po(\lambda) = \{po(i; \lambda), i = 0, 1, \dots\}$ , with  $po(i; \lambda) = \lambda^i e^{-\lambda} / i!$ , we have

$$T_\alpha(po(\lambda)) = po(\alpha\lambda).$$

For the binomial pmf  $bi(n, p) = \{bi(i; n, p), i = 0, \dots, n\}$ , with  $bi(i; n, p) = \binom{n}{i} p^i (1-p)^{n-i}$ , we have

$$T_\alpha(bi(n, p)) = bi(n, \alpha p).$$

There also holds the semigroup relation ([20])

$$T_\alpha(T_\beta(f)) = T_{\alpha\beta}(f). \quad (1)$$

Such relations hint at the role of thinning as a discrete counterpart of scaling, which we explore from an information theoretic perspective, as did [20].

The  $n$ -th convolution of  $f$ , denoted as  $f^{*n}$ , is the pmf of  $\sum_{i=1}^n Y_i$  where  $Y_i$ ’s are i.i.d. with pmf  $f$ . It is easy to show that thinning and convolution operations commute, i.e.,

$$T_\alpha(f^{*n}) = (T_\alpha(f))^{*n}. \quad (2)$$

Using the notions of thinning and convolution, we state the following version of the law of small numbers considered by [15], [17]. As usual, for two pmfs  $f$  and  $g$ , the entropy of  $f$  is defined as  $H(f) = -\sum_i f_i \log(f_i)$ , and the relative entropy between  $f$  and  $g$  is defined as  $D(f|g) = \sum_i f_i \log(f_i/g_i)$ . It is understood that  $D(f|g) = \infty$  if the support of  $f$ ,  $\text{supp}(f) = \{i : f_i > 0\}$ , is not a subset of  $\text{supp}(g)$ . We frequently consider the relative entropy between a pmf  $f$  and  $po(\lambda)$ , where  $\lambda$  is the mean of  $f$ ; we denote

$$D(f) = D(f|po(\lambda))$$

for convenience.

**Theorem 1:** Let  $f$  be a pmf on  $\mathbf{Z}_+$  with mean  $\lambda < \infty$ . Then, as  $n \rightarrow \infty$ ,

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- 1)  $T_{1/n}(f^{*n})$  tends to  $po(\lambda)$  pointwise;
- 2)  $H(T_{1/n}(f^{*n})) \rightarrow H(po(\lambda))$ ;
- 3) if  $D(T_{1/n}(f^{*n}))$  ever becomes finite, then it tends to zero.

Part 1) of Theorem 1 is proved by Harremoës et al. [15], who also present a proof of Part 3) assuming  $D(f) < \infty$ . The current, slightly more general form of Part 3) is reminiscent of Barron's work [4] on the CLT. In Section II we present a short proof of Part 3). We also note that Part 2), which is stated in [16] with a stronger assumption, can be deduced from 1) directly.

A major goal of this work is to establish monotonicity properties in Theorem 1. We show that, in Part 3) of Theorem 1, the relative entropy never increases (Theorem 2), and, assuming  $f$  is *ultra-log-concave* (see Definition 2), in Part 2) of Theorem 1, the entropy never decreases (Theorem 3). Both Theorems 2 and 3 can be regarded as discrete analogues of the monotonicity of entropy in the CLT ([2]), with thinning playing the role of scaling. (Unlike the CLT case, here monotonicity of the entropy and that of the relative entropy are not equivalent.) We begin with monotonicity of the relative entropy.

**Theorem 2:** If  $f$  is a pmf on  $\mathbf{Z}_+$  with a finite mean, then  $D(T_{1/n}(f^{*n}))$  decreases on  $n = 1, 2, \dots$

The proof of Theorem 2 uses two Lemmas, which are of interest by themselves. These deal with the behavior of relative entropy under thinning (Lemma 1) and convolution (Lemma 2) respectively. Lemma 1 is proved in Section III, where we also note its close connection with modified logarithmic Sobolev inequalities (Bobkov and Ledoux [6]; Wu [36]) for the Poisson distribution.

**Lemma 1 (The Thinning Lemma):** Let  $f$  be a pmf on  $\mathbf{Z}_+$  with a finite mean. Then

$$D(T_\alpha(f)) \leq \alpha D(f), \quad 0 < \alpha < 1.$$

An equivalent statement is that  $\alpha^{-1}D(T_\alpha(f))$  increases in  $\alpha \in (0, 1]$ , in view of the semigroup property (1).

Combined with a data processing argument, Lemma 1 can be used to show that the relative entropy is monotone along power-of-two iterates in Theorem 2. To prove Theorem 2 fully, however, we need the following convolution result, which may be seen as a "strengthened data processing inequality." Lemma 2 is proved in Section IV.

**Lemma 2 (The Convolution Lemma):** If  $f$  is a pmf on  $\mathbf{Z}_+$  with a finite mean, then  $(1/n)D(f^{*n})$  decreases in  $n$ .

The main difference in the development here, compared with the CLT case, is that we need to consider the effect of both thinning and convolution. In the CLT case, the monotonicity of entropy can be obtained from one general convolution inequality for the Fisher information ([2], [27]). Nevertheless, the proofs of Lemmas 1 and 2 (Lemma 2 in particular) somewhat parallel the CLT case. We first express the desired divergence quantity as an integral via a de Bruijn type identity ([34], [5], [4]), and then analyze the monotonicity property of the integrand; see Sections III and IV for details.

Once we have Lemmas 1 and 2, Theorem 2 is quickly established.

*Proof of Theorem 2:* Applying Lemma 1 with  $\alpha = (n-1)/n$ , and using (1), we get ( $n \geq 2$ )

$$\frac{n}{n-1}D(T_{1/n}(f^{*n})) \leq D(T_{1/(n-1)}(f^{*n})).$$

Lemma 2 and (2) then yield

$$D(T_{1/(n-1)}(f^{*n})) \leq \frac{n}{n-1}D(T_{1/(n-1)}(f^{*(n-1)}))$$

and the claim follows.  $\blacksquare$

By a different analysis, we also establish the monotonicity of  $H(T_{1/n}(f^{*n}))$ , under the assumption that  $f$  is *ultra-log-concave*.

**Definition 2:** A nonnegative sequence  $u = \{u_i, i \in \mathbf{Z}_+\}$  is called *log-concave*, if the support of  $u$  is an interval of consecutive integers, and  $u_i^2 \geq u_{i-1}u_{i+1}$  for all  $i > 0$ . A pmf  $f$  is *ultra-log-concave*, or ULC, if the sequence  $i!f_i, i \in \mathbf{Z}_+$ , is log-concave.

Equivalently,  $f$  is ULC if  $if_i/f_{i-1}$  decreases in  $i$ . It is clear that ultra-log-concavity implies log-concavity. Examples of ULC pmfs include the Poisson and the binomial. More generally, the pmf of  $\sum_{i=1}^n X_i$  is ULC if  $X_i$ 's are independent (not necessarily identically distributed) Bernoulli random variables.

The monotonicity of entropy is stated as follows.

**Theorem 3:** If  $f$  is ULC, then  $H(T_{1/n}(f^{*n}))$  increases monotonically on  $n = 1, 2, \dots$

An example ([14] [41]) is when  $f$  is a Bernoulli with parameter  $p$ , in which case  $T_{1/n}(f^{*n}) = bi(n, p/n)$ . In other words, both the entropy and the relative entropy are monotone in the classical binomial-to-Poisson convergence.

It should not be surprising that we make the ULC assumption; the situation is similar to that of a Markov chain with homogeneous transition probabilities ([10], Chapter 4): relative entropy always decreases, but entropy does not increase without additional assumptions. The ULC assumption is natural in Theorem 3 because ULC distributions with the same mean  $\lambda$  form a natural class in which the  $Po(\lambda)$  distribution has maximum entropy [20]. In fact, if we reverse the ULC assumption (see Theorem 7 for the precise statement), but still assume that  $f$  is log-concave, then  $H(T_{1/n}(f^{*n}))$  decreases monotonically. Theorems 3 and 7 are proved in Section VI. The starting point in these proofs is a general result (Lemma 4) that relates entropy comparison to comparing the expectations of convex functions. This entails a rather detailed analysis of the convex order (to be defined in Section V) between the relevant distributions.

As a simple example, Fig. 1 displays the values of

$$\begin{aligned} d(n) &= D(T_{1/n}(f^{*n})), & t(n) &= nD(T_{1/n}(f)), \\ r(n) &= n^{-1}D(f^{*n}), & h(n) &= H(T_{1/n}(f^{*n})) \end{aligned}$$

for  $f = bi(2, 1/2)$  and  $n = 1, \dots, 10$ . The monotone patterns of  $d(n)$ ,  $t(n)$ ,  $r(n)$  and  $h(n)$  illustrate Theorem 2, Lemma 1, Lemma 2, and Theorem 3 respectively.

Besides monotonicity, an equally interesting problem is the rate of convergence. In Section VII we show that, if  $f$  is ULC or has finite support, then  $D(T_{1/n}(f^{*n})) = O(n^{-2})$ ,  $n \rightarrow \infty$ . This complements certain bounds obtained by Harremoës et al. [15], [16], [17]. Different tools contribute to this  $O(n^{-2})$  rate.

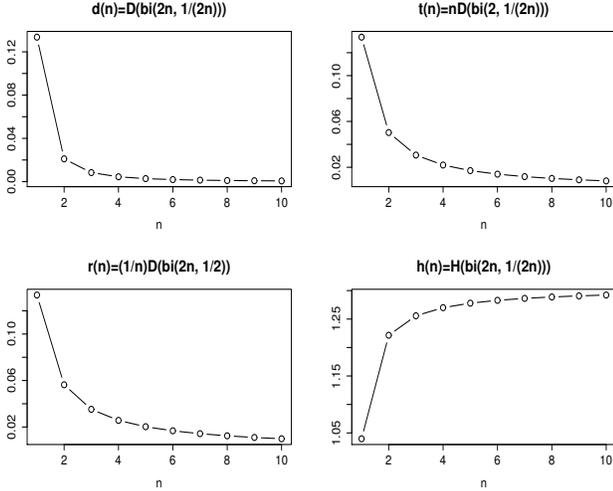


Fig. 1. Values of  $d(n)$ ,  $t(n)$ ,  $r(n)$  and  $h(n)$  for  $n = 1, \dots, 10$ .

For ULC distributions we use stochastic orders as in Section VI; for distributions with finite support, we simply analyze the *scaled Fisher information* ([24], [28]). We conclude with a discussion on possible extensions and refinements (of Theorem 2 in particular) in Section VIII.

## II. THE CONVERGENCE THEOREM

This section deals with Theorem 1. Part 1) of Theorem 1 is proved in [15]. Part 2) can be quickly proved as follows. Part 1) and Fatou's lemma yield

$$\liminf_{n \rightarrow \infty} H(T_{1/n}(f^{*n})) \geq H(po(\lambda)).$$

Let  $g$  denote the pmf of a geometric( $p$ ) distribution, i.e.,  $g_i = p(1-p)^i$ ,  $i = 0, 1, \dots$ ,  $0 < p < 1$ . By the lower-semicontinuity property of relative entropy,

$$\liminf_{n \rightarrow \infty} D(T_{1/n}(f^{*n})|g) \geq D(po(\lambda)|g). \quad (3)$$

Since the mean of  $T_{1/n}(f^{*n})$  is  $\lambda$  for all  $n$ , (3) simplifies to

$$\limsup_{n \rightarrow \infty} H(T_{1/n}(f^{*n})) \leq H(po(\lambda))$$

and Part 2) is proved.

Our proof of Part 3) uses convexity arguments that also yield some interesting intermediate results. In Propositions 1–3 let  $X_1, X_2, \dots$ , be i.i.d. with pmf  $f$ .

**Proposition 1:** For any  $\alpha \in (0, 1]$ ,  $D(T_\alpha(f)) < \infty$  if and only if  $EX_1 \log X_1 < \infty$  (as usual  $0 \log 0 = 0$ ).

*Proof:* Let us consider  $\alpha = 1$  first. Note that  $H(f)$  is finite since the mean of  $f$  is finite. We have

$$D(f) = \sum_{i \geq 0} f_i \log(i!) - \lambda \log(\lambda) + \lambda - H(f).$$

Thus  $D(f) < \infty$  if and only if  $\sum_{i \geq 0} f_i \log(i!)$  converges, which, by Stirling's formula, is equivalent to  $EX_1 \log X_1 < \infty$ .

For general  $\alpha \in (0, 1]$ , let  $Y|X_1 \sim \text{Bi}(X_1, \alpha)$ . By the preceding argument  $D(T_\alpha(f)) < \infty$  if and only if  $EY \log Y < \infty$ . However

$$E[\alpha X_1 \log(\alpha X_1)] \leq EY \log Y \leq EX_1 \log X_1$$

where the lower bound holds by Jensen's inequality applied to the convex function  $y \log y$ . Thus  $EY \log Y < \infty$  is also equivalent to  $EX_1 \log X_1 < \infty$ . ■

A consequence of Proposition 1 is that, in Part 3),

$$D(T_{1/n}(f^{*n})) < \infty \iff E\bar{X}_n \log \bar{X}_n < \infty.$$

Here and in Propositions 2 and 3 below,  $\bar{X}_n = (1/n) \sum_{i=1}^n X_i$ .

**Proposition 2:** For  $n \geq 1$ ,

$$D(T_{1/n}(f^{*n})) \leq \frac{\lambda}{n} + E\bar{X}_n \log \frac{\bar{X}_n}{\lambda}.$$

The proof of Proposition 2 borrows an idea of [15] and is included in the Appendix; see [17], Proposition 15, for a subsequent refinement.

**Proposition 3:** Define  $l_n = E\bar{X}_n \log(\bar{X}_n/\lambda)$ . Then, as  $n \uparrow \infty$ ,  $l_n$  decreases to zero if it is finite for some  $n$ .

*Proof:* By Jensen's inequality (applied to the convex function  $x \log x$ ), we have  $l_n \geq 0$ . Noting  $\bar{X}_n = E\bar{X}_{n-1}|\bar{X}_n$ , we use Jensen's inequality for the conditional expectation to get

$$l_n \leq EE[\bar{X}_{n-1} \log(\bar{X}_{n-1}/\lambda)|\bar{X}_n] = l_{n-1}.$$

(Essentially we are proving  $\bar{X}_n \leq_{cx} \bar{X}_{n-1}$  where  $\leq_{cx}$  denotes the convex order; see [31]. Section V contains a brief introduction to several stochastic orders.) Thus  $l_n \downarrow l_\infty$ , say, with  $l_\infty \geq 0$ .

We show  $l_\infty = 0$ , assuming  $l_k < \infty$  for some  $k$ . By symmetry  $l_n = E\bar{X}_k \log(\bar{X}_n/\lambda)$ ,  $n \geq k$ . We may use this and Jensen's inequality (applied to the concave function  $\log x$ ) to obtain

$$\begin{aligned} l_n &\leq E\bar{X}_k \log \frac{E\bar{X}_n|\bar{X}_k}{\lambda} \\ &= E\bar{X}_k \log \frac{k\bar{X}_k + (n-k)\lambda}{n\lambda}. \end{aligned} \quad (4)$$

However,

$$\bar{X}_k \log \frac{k\bar{X}_k + (n-k)\lambda}{n\lambda} \leq \bar{X}_k \max \left\{ 0, \log \frac{\bar{X}_k}{\lambda} \right\},$$

and the right hand side has a finite expectation since  $l_k < \infty$ . Letting  $n \rightarrow \infty$  in (4) and using Fatou's lemma we obtain

$$l_\infty \leq E\bar{X}_k \log \frac{\lambda}{\lambda} = 0$$

which forces  $l_\infty = 0$ . ■

Part 3) is then a direct consequence of Propositions 1–3.

## III. LEMMA 1 AND A MODIFIED LOGARITHMIC SOBOLEV INEQUALITY

For any pmfs  $\tilde{g}$  and  $g$  on  $\mathbf{Z}_+$ , we have

$$D(T_\alpha(\tilde{g})|T_\alpha(g)) \leq D(\tilde{g}|g). \quad (5)$$

This is a special case of a general result on the decrease of relative entropy along a Markov chain (see [10], Chapter 4). It follows from (5) and the semigroup property (1) that, in the setting of Lemma 1,  $D(T_\alpha(f))$  increases in  $\alpha$ . This is however not strong enough to prove Lemma 1 yet.

Let us recall the *size-biasing* operation, which often appears in Poisson approximation problems.

**Definition 3:** For a pmf  $f$  on  $\mathbf{Z}_+$  with mean  $\lambda > 0$ , the sized-biased pmf, denoted by  $S(f)$ , is defined on  $\mathbf{Z}_+$  as

$$S(f) = \{(i+1)f_{i+1}/\lambda, i = 0, 1, \dots\}.$$

The formulas  $S(po(\lambda)) = po(\lambda)$  and  $S(bi(n, p)) = bi(n-1, p)$  are readily verified. Moreover, size-biasing and thinning operations commute, i.e.,

$$T_\alpha(S(f)) = S(T_\alpha(f)). \quad (6)$$

Key to the proof of Lemma 1 is the following identity; see [20] for related calculations.

**Lemma 3:** Let  $f = \{f_i, i \geq 0\}$  be a pmf on  $\mathbf{Z}_+$  with mean  $\lambda \in (0, \infty)$ , and assume that the support of  $f$  is finite. Then

$$\frac{dD(T_\alpha(f))}{d\alpha} = \lambda D(T_\alpha(S(f))|T_\alpha(f)), \quad \alpha \in (0, 1). \quad (7)$$

*Proof:* Write  $g = T_\alpha(f)$  for convenience, i.e.,

$$g_i = \sum_{j \geq i} f_j bi(i; j, \alpha).$$

Using the identity (see [20], Proposition 3.6)

$$\frac{dg_i}{d\alpha} = \frac{ig_i - (i+1)g_{i+1}}{\alpha},$$

we get

$$\begin{aligned} \frac{dD(g)}{d\alpha} &= \sum_{i \geq 0} \frac{ig_i - (i+1)g_{i+1}}{\alpha} \log \frac{g_i}{po(i; \alpha\lambda)} \\ &= \sum_{i \geq 1} \frac{ig_i}{\alpha} \left[ \log \frac{g_i}{po(i; \alpha\lambda)} - \log \frac{g_{i-1}}{po(i-1; \alpha\lambda)} \right] \\ &= \lambda D(S(g)|g) \\ &= \lambda D(T_\alpha(S(f))|T_\alpha(f)) \end{aligned}$$

where we use summation by parts in the second step, and (6) in the fourth. All sums are finite sums since  $f$  has finite support. ■

**Remark.** The assumption that  $f$  has finite support does not appear to impose a serious limit on the applicability of Lemma 3. Of course, it would be good to see this assumption relaxed.

*Proof of Lemma 1:* Let us first assume that  $f$  has finite support. Then  $D(T_\alpha(f))$  is finite and continuous on  $\alpha \in [0, 1]$ . Applying (5) and (1) to the right hand side of (7), we deduce that  $dD(T_\alpha(f))/d\alpha$  increases on  $\alpha \in (0, 1)$ . Thus  $D(T_\alpha(f))$  is convex on  $\alpha \in [0, 1]$ , and the claim follows. For general  $f$ , we construct a sequence of pmfs  $f^{(k)} = \{f_i^{(k)}, i \geq 0\}$ ,  $k = 1, 2, \dots$ , by truncation. In other words, let  $f_i^{(k)} = c_k f_i$ ,  $i = 0, \dots, k$ , where  $c_k = (\sum_{i \leq k} f_i)^{-1}$ , and  $f_i^{(k)} = 0$ ,  $i > k$ . Assume  $D(f) < \infty$  without loss of generality. Then  $T_\alpha(f^{(k)})$  tends to  $T_\alpha(f)$  pointwise as  $k \rightarrow \infty$ . It is also easy to show

$$D(f^{(k)}) \rightarrow D(f), \quad k \rightarrow \infty.$$

Thus, by the finite-support result and the lower-semicontinuity property of the relative entropy, we have

$$\begin{aligned} D(T_\alpha(f)) &\leq \liminf_{k \rightarrow \infty} D(T_\alpha(f^{(k)})) \\ &\leq \liminf_{k \rightarrow \infty} \alpha D(f^{(k)}) \\ &= \alpha D(f) \end{aligned}$$

as required. ■

For two pmfs  $f$  and  $g$  on  $\mathbf{Z}_+$  with finite means, the data-processing inequality ([10]) gives (\* denotes convolution)

$$D(T_\alpha(f) * T_\beta(g)) \leq D(T_\alpha(f)) + D(T_\beta(g)) \quad (8)$$

where  $\alpha, \beta \in [0, 1]$ . By Lemma 1, we have

$$D(T_\alpha(f) * T_\beta(g)) \leq \alpha D(f) + \beta D(g).$$

This is enough to prove Theorem 2 in the special case of power-of-two iterates, i.e.,  $D(T_{1/n}(f^{*n}))$  decreases on  $n = 2^k$ ,  $k = 0, 1, \dots$ . To establish Theorem 2 fully, we need a convolution inequality stronger than (8), namely Lemma 2; Section IV contains the details.

A result closely related to Lemma 1 is Theorem 4, which was proved by Wu ([36], Eqn. 0.6) using advanced stochastic calculus tools (see [6], [8], [9] for related work). Our proof of Theorem 4, based on convexity, is similar in spirit to those given by [8], [9]; the use of thinning appears new.

**Theorem 4 ([36]):** For a pmf  $f$  on  $\mathbf{Z}_+$  with mean  $\lambda \in (0, \infty)$  we have

$$D(f) \leq \lambda D(S(f)|f).$$

*Proof:* Let us assume the support of  $f$  is finite. The convexity of  $h(\alpha) = D(T_\alpha(f))$  implies  $h'(\alpha) \geq h(\alpha)/\alpha$  for all  $\alpha \in (0, 1)$ . If  $D(S(f)|f) < \infty$  then  $\text{supp}(f)$  is an interval of consecutive integers including zero. We may let  $\alpha \rightarrow 1$  in (7) and obtain

$$\lambda D(S(f)|f) = \lim_{\alpha \uparrow 1} h'(\alpha) \geq h(1) = D(f).$$

When the support of  $f$  is not finite, an argument similar to the one for Lemma 1 applies. ■

Theorem 4 sharpens a modified logarithmic Sobolev inequality originally obtained by Bobkov and Ledoux [6].

**Corollary 1 ([6], Corollary 4):** In the setting of Theorem 4, assume that  $f_i > 0$  for all  $i \in \mathbf{Z}_+$ . Then

$$D(f) \leq \lambda \chi^2(S(f), f) \quad (9)$$

where  $\chi^2(S(f), f) = \sum_i f_i ((S(f))_i / f_i - 1)^2$ .

The inequality (9) follows from Theorem 4 and the well-known inequality between the relative entropy and the  $\chi^2$  distance. For an application of (9) to Poisson approximation bounds, see [24].

#### IV. RELATIVE ENTROPY UNDER CONVOLUTION

This section establishes Lemma 2. The starting point is an easily verified decomposition formula (Proposition 4). Proposition 4 was used by Madiman et al. [28] to derive a convolution inequality ([28], Theorem III) for the scaled Fisher information, which is  $\lambda \chi^2(S(f), f)$  as in (9). Here we obtain

a monotonicity result (Corollary 2) for the relative entropy  $D(S(f^{*n})|f^{*n})$ , which is instrumental in the proof of Lemma 2.

**Proposition 4 ([28], Eqn. 14):** Let  $q^{(i)}$  be pmfs on  $\mathbf{Z}_+$  with finite means  $\lambda_i$ ,  $i = 1, \dots, n$ , respectively ( $n \geq 2$ ). Define  $q = q^{(1)} * \dots * q^{(n)}$  and  $q^{(-i)} = q^{(1)} * \dots * q^{(i-1)} * q^{(i+1)} * \dots * q^{(n)}$  (i.e.,  $q^{(i)}$  is left out),  $i = 1, \dots, n$ . Then there holds

$$S(q) = \sum_{i=1}^n \beta_i q^{(i)} * S(q^{(-i)})$$

where  $\beta_i = (1 - \lambda_i / \sum_{j=1}^n \lambda_j) / (n - 1)$ . (In statistical terms, we have a mixture representation of  $S(q)$ .)

**Proposition 5:** In the setting of Proposition 4 we have

$$D(q|S(q)) \leq \sum_{i=1}^n \beta_i D(q^{(-i)}|S(q^{(-i)})); \quad (10)$$

$$D(S(q)|q) \leq \sum_{i=1}^n \beta_i D(S(q^{(-i)})|q^{(-i)}). \quad (11)$$

*Proof:* We prove (10); the same argument applies to (11). By convexity, Proposition 4 yields

$$D(q|S(q)) \leq \sum_{i=1}^n \beta_i D(q|q^{(i)} * S(q^{(-i)})).$$

However, since  $q = q^{(i)} * q^{(-i)}$  for each  $i$ , we have

$$D(q|q^{(i)} * S(q^{(-i)})) \leq D(q^{(-i)}|S(q^{(-i)}))$$

by data processing, and the claim follows.  $\blacksquare$

Corollary 2 corresponds to the case of identical  $q^{(i)}$ 's in Proposition 5.

**Corollary 2:** For any pmf  $f$  on  $\mathbf{Z}_+$  with mean  $\lambda \in (0, \infty)$ , both  $D(S(f^{*n})|f^{*n})$  and  $D(f^{*n}|S(f^{*n}))$  decrease in  $n$ .

*Proof of Lemma 2:* Let us assume that  $f$  has finite support first. We have (7) in the integral form

$$\begin{aligned} \frac{1}{n} D(f^{*n}) &= \lambda \int_0^1 D(T_\alpha(S(f^{*n}))|T_\alpha(f^{*n})) d\alpha \\ &= \lambda \int_0^1 D(S((T_\alpha(f))^{*n})|(T_\alpha(f))^{*n}) d\alpha \end{aligned} \quad (12)$$

where (12) holds by the commuting relations (6) and (2). By Corollary 2, the integrand in (12) decreases in  $n$  for each  $\alpha$ . Thus  $(1/n)D(f^{*n})$  decreases in  $n$  as claimed. For general  $f$ , we again use truncation. Let  $f^{(k)}$  and  $c_k$  be defined as in the proof of Lemma 1. For  $n \geq 2$  let  $g = f^{*n}$ , and similarly let  $g^{(k)}$  denote the  $n$ th convolution of  $f^{(k)}$ . Then  $g^{(k)}$  tends to  $g$  pointwise, and the mean of  $g^{(k)}$  tends to that of  $g$ . Assume  $D(g) < \infty$ , which amounts to  $\sum_i g_i \log(i!) < \infty$ . The argument for Part 2) of Theorem 1 shows

$$H(g^{(k)}) \rightarrow H(g), \quad k \rightarrow \infty. \quad (13)$$

We also have the inequality  $g_i^{(k)} \leq c_k^n g_i$ , because  $f_i^{(k)} \leq c_k f_i$ ,  $i = 0, 1, \dots$ . Since  $c_k \rightarrow 1$  as  $k \rightarrow \infty$ , we may apply dominated convergence to obtain

$$\sum_i g_i^{(k)} \log(i!) \rightarrow \sum_i g_i \log(i!), \quad k \rightarrow \infty,$$

which, taken together with (13), shows

$$D(g^{(k)}) \rightarrow D(g), \quad k \rightarrow \infty.$$

The finite-support result and the lower-semicontinuity property of relative entropy then yield

$$\frac{1}{n+1} D(f^{*(n+1)}) \leq \frac{1}{n} D(f^{*n})$$

as in the proof of Lemma 1.  $\blacksquare$

A generalization of Lemma 2 is readily obtained if we use Proposition 5 rather than Corollary 2 in the above argument.

**Theorem 5:** In the setting of Proposition 4,

$$D(q) \leq \frac{1}{n-1} \sum_{i=1}^n D(q^{(-i)}).$$

Theorem 5 strengthens the usual data processing inequality

$$D(q) \leq \sum_{i=1}^n D(q^{(i)})$$

in the same way that the entropy power inequality of Artstein et al. [2] strengthens Shannon's classical entropy power inequality.

**Remark.** A by-product of Corollary 2 is that the divergence quantities

$$\begin{aligned} h_n &= D(T_{1/n}(f^{*n})|S(T_{1/n}(f^{*n}))) \quad \text{and} \\ \tilde{h}_n &= D(S(T_{1/n}(f^{*n}))|T_{1/n}(f^{*n})) \end{aligned}$$

also decrease in  $n$ . Indeed we have

$$\begin{aligned} h_n &= D(T_{1/n}(f^{*n})|T_{1/n}(S(f^{*n}))) \\ &\leq D(T_{1/(n-1)}(f^{*n})|T_{1/(n-1)}(S(f^{*n}))) \end{aligned} \quad (14)$$

$$\begin{aligned} &= D((T_{1/(n-1)}(f))^{*n}|S((T_{1/(n-1)}(f))^{*n})) \\ &\leq h_{n-1} \end{aligned} \quad (15)$$

where (5) is used in (14), Corollary 2 is used in (15), and the commuting relations (6) and (2) are applied throughout. The proof for  $\tilde{h}_n$  is the same. These monotonicity statements complement Theorem 2.

## V. STOCHASTIC ORDERS AND MAJORIZATION

The proof of the monotonicity of entropy (Theorem 3) involves several notions of stochastic orders which we briefly introduce; see Shaked and Shanthikumar [31] for further results.

**Definition 4:** For two random variables  $X$  and  $Y$  with pmfs  $f$  and  $g$  respectively,

- $X$  is smaller than  $Y$  in the *usual stochastic order*, written as  $X \leq_{st} Y$ , if  $\Pr(X > c) \leq \Pr(Y > c)$  for all  $c$ ;
- $X$  is smaller than  $Y$  in the *convex order*, written as  $X \leq_{cx} Y$ , if  $E\phi(X) \leq E\phi(Y)$  for every convex function  $\phi$  such that the expectations exist;
- $X$  is log-concave relative to  $Y$ , written as  $X \leq_{lc} Y$ , if i) both  $\text{supp}(f)$  and  $\text{supp}(g)$  are intervals of consecutive integers, ii)  $\text{supp}(f) \subset \text{supp}(g)$ , and iii)  $\log(f_i/g_i)$  is concave on  $\text{supp}(f)$ .

We use  $\leq_{st}$ ,  $\leq_{cx}$ ,  $\leq_{lc}$  with the pmfs as well as the random variables. In general,  $f \leq_{st} g$  if we can construct random

variables  $X$  and  $Y$  with pmfs  $f$  and  $g$  respectively on the same probability space such that  $X \leq Y$  almost surely. Examples include

$$bi(n, p) \leq_{st} bi(n+1, p), \quad bi(n, p) \leq_{st} bi(n, \tilde{p}), \quad p \leq \tilde{p}.$$

In contrast,  $\leq_{cx}$  compares variability. A classical example (Hoeffding [19]) is

$$bi(n, \lambda/n) \leq_{cx} bi(n+1, \lambda/(n+1)), \quad 0 \leq \lambda \leq n.$$

Another example mentioned in Section II is  $\bar{X}_n \leq_{cx} \bar{X}_{n-1}$  where  $\bar{X}_n = (1/n) \sum_{i=1}^n X_i$  for i.i.d.  $X_i$ 's with a finite mean. The log-concavity order  $\leq_{lc}$  is also useful in our context; for example,  $f$  being ULC can be written as  $f \leq_{lc} po(\lambda)$ ,  $\lambda > 0$ . (The actual value of  $\lambda$  is immaterial.)

We also need the concepts of *majorization* and *Schur concavity*.

**Definition 5:** A real vector  $\mathbf{b} = (b_1, \dots, b_n)$  is said to majorize  $\mathbf{a} = (a_1, \dots, a_n)$ , written as  $\mathbf{a} < \mathbf{b}$ , if

- $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ , and
- $\sum_{i=k}^n a_{(i)} \leq \sum_{i=k}^n b_{(i)}$ ,  $k = 2, \dots, n$ , where  $a_{(1)} \leq \dots \leq a_{(n)}$  and  $b_{(1)} \leq \dots \leq b_{(n)}$  are  $(a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$  arranged in increasing order, respectively.

A function  $\phi(\mathbf{a})$  symmetric in the coordinates of  $\mathbf{a} = (a_1, \dots, a_n)$  is said to be *Schur concave*, if

$$\mathbf{a} < \mathbf{b} \implies \phi(\mathbf{a}) \geq \phi(\mathbf{b}).$$

As is well-known, if pmfs  $f$  and  $g$  on  $\{0, \dots, n\}$  (viewed as vectors of the respective probabilities) satisfy  $f < g$ , then  $H(f) \geq H(g)$ . In other words  $H(f)$  is a Schur concave function of  $f$ . Further properties and various applications of these two notions can be found in Hardy et al. [13] and Marshall and Olkin [29].

## VI. MONOTONICITY OF THE ENTROPY

This section proves Theorem 3. We state a key lemma that can be traced back to Karlin and Rinott [23].

**Lemma 4:** Let  $f$  and  $g$  be pmfs on  $\mathbf{Z}_+$  such that  $f \leq_{cx} g$  and  $g$  is log-concave. Then

$$H(f) + D(f|g) \leq H(g).$$

In particular  $H(f) \leq H(g)$  with equality only if  $f = g$ .

Although Lemma 4 follows almost immediately from the definitions (hence the proof is omitted), it is a useful tool in several entropy comparison contexts ([23], [41], [38], [39]). Effectively, Lemma 4 reduces entropy comparison to two (often easier) problems: i) establishing a log-concavity result, and ii) comparing the expectations of convex functions. A modification of Lemma 4 is used by [41] to give a short and unified proof of the main theorems of [20] and [37] concerning the maximum entropy properties of the Poisson and binomial distributions. We quote Johnson's result. Further extensions to compound distributions can be found in [22], [39].

**Theorem 6:** If a pmf  $f$  on  $\mathbf{Z}_+$  is ULC with mean  $\lambda$ , then  $H(f) \leq H(po(\lambda))$ , with equality only if  $f = po(\lambda)$ .

To apply Lemma 4 to our problem, we show that, in the setting of Theorem 3,

$$T_{1/(n-1)}(f^{*(n-1)}) \leq_{cx} T_{1/n}(f^{*n}). \quad (16)$$

In a sense, (16) means that  $T_{1/n}(f^{*n})$  becomes more and more "spread out" as  $n$  increases. On the other hand, it can be shown that  $T_{1/n}(f^{*n})$  is log-concave for all  $n$ . Indeed,  $f$  is ULC and hence log-concave. It is well-known that convolution preserves log-concavity. That thinning preserves log-concavity is sometimes known as Brenti's criterion [7] in the combinatorics literature. Thus  $T_{1/n}(f^{*n})$  remains log-concave. Actually, since  $f$  is ULC, there holds the stronger relation

$$T_{1/n}(f^{*n}) \leq_{lc} po(\lambda). \quad (17)$$

Relation (17) follows from i) if  $f$  is ULC then so is  $f^{*n}$  (Liggett [26]) and ii) if  $f$  is ULC then so is  $T_\alpha(f)$  (Johnson [20], Proposition 3.7).

The core of the proof of Theorem 3 is proving (16). The notions of majorization and Schur concavity briefly reviewed in Section V are helpful in formulating a more general (and easier to handle) version of (16).

**Proposition 6:** Let  $Y_1, \dots, Y_n$  be i.i.d. random variables on  $\mathbf{Z}_+$  with an ultra-log-concave pmf  $f$ . Conditional on the  $Y_i$ 's, let  $Z_i$ ,  $i = 1, \dots, n$ , be independent  $\text{Bi}(Y_i, p_i)$  random variables respectively, where  $p_1, \dots, p_n \in [0, 1]$ . Let  $\phi$  be a convex function on  $\mathbf{Z}_+$ . Then  $E\phi(\sum_{i=1}^n Z_i)$  is a Schur concave function of  $(p_1, \dots, p_n)$  on  $[0, 1]^n$ .

The proof of Proposition 6, somewhat technical, is collected in the Appendix.

*Proof of (16):* Noting that

$$(1/n, \dots, 1/n) < (1/(n-1), \dots, 1/(n-1), 0)$$

the claim follows from Proposition 6 and the definition of Schur-concavity. ■

*Proof of Theorem 3:* Based on (16) and (17), the claim follows by applying Lemma 4 with  $T_{1/(n-1)}(f^{*(n-1)})$  in place of  $f$ , and  $T_{1/n}(f^{*n})$  in place of  $g$ . ■

**Remark.** Theorem 3 resembles the semigroup argument of Johnson [20] in that both are statements of "entropy increasing to the maximum," and both involve convolution and thinning operations. The difference is that [20] considers convolution with a Poisson while we study the self-convolution  $f^{*n}$ .

As mentioned in Section I, if we reverse the ULC assumption (but still assume log-concavity), then the conclusion of Theorem 3 is also reversed.

**Theorem 7:** Let  $f$  be a pmf on  $\mathbf{Z}_+$  with mean  $\lambda$ . Assume  $f$  is log-concave, and assume  $po(\lambda) \leq_{lc} f$ . Then  $H(T_{1/n}(f^{*n}))$  decreases in  $n$ .

Theorem 7 extends a minimum entropy result that parallels Theorem 6.

**Proposition 7 ([41]):** The  $\text{Po}(\lambda)$  distribution achieves minimum entropy among all pmfs  $f$  with mean  $\lambda$  such that  $f$  is log-concave and  $po(\lambda) \leq_{lc} f$ .

An example of Theorem 7, also noted in [41], is when  $f$  is a geometric( $p$ ) pmf, in which case  $T_{1/n}(f^{*n}) = nb(n, n/(n-1+1/p))$ . (Here  $nb(n, p)$  denotes the negative binomial pmf with parameters  $(n, p)$ , i.e.,  $nb(n, p) = \binom{n+i-1}{i} p^n (1-p)^i$ ,  $i = 0, 1, \dots$ .) In other words, the negative-binomial-to-Poisson convergence is monotone in entropy (as long as the first parameter of the negative binomial is at least 1).

The proof of Theorem 7 parallels that of Theorem 3. In place of (16) we have

$$T_{1/n}(f^{*n}) \leq_{cx} T_{1/(n-1)}(f^{*(n-1)})$$

assuming  $po(\lambda) \leq_{lc} f$ . The proof of (16) applies after reversing the direction of  $\leq$  in the relevant places. As noted before, since  $f$  is log-concave,  $T_{1/n}(f^{*n})$  is log-concave for all  $n$ . Thus Theorem 7 follows from Lemma 4 as does Theorem 3.

Incidentally, we have

$$po(\lambda) \leq_{lc} f \implies po(\lambda) \leq_{lc} T_{1/n}(f^{*n}), \quad (18)$$

which is a reversal of (17). To prove (18), we note that, according to a result of Davenport and Pólya [11],  $po(\lambda) \leq_{lc} f$  implies  $po(\lambda) \leq_{lc} f^{*n}$ . By a slight modification of the argument of Johnson ([20], Proposition 3.7), we can also show that  $po(\lambda) \leq_{lc} f$  implies  $po(\lambda) \leq_{lc} T_\alpha(f)$  (details omitted); thus (18) holds.

## VII. RATE OF CONVERGENCE

Assuming that  $f$  is a pmf on  $\mathbf{Z}_+$  with mean  $\lambda$  and variance  $\sigma^2 < \infty$ , Harremoës et al. ([15], Corollary 9) show that

$$D(T_{1/n}(f^{*n})) \leq \frac{\lambda}{n} + \frac{\sigma^2}{n\lambda}.$$

That is, the relative entropy converges at a rate of (at least)  $O(n^{-1})$ . We aim to improve this to  $O(n^{-2})$  under some natural assumptions. The  $O(n^{-2})$  rate is perhaps not surprising since, in the binomial case ([18]),

$$D(bi(n, \lambda/n)) = O(n^{-2}), \quad n \rightarrow \infty. \quad (19)$$

We first use the stochastic orders  $\leq_{cx}$  and  $\leq_{lc}$  to extend (19) to ULC distributions. Note that, while this paper is under review, some related  $O(n^{-2})$  bounds have been obtained by [17].

**Theorem 8:** If  $f$  is ULC on  $\mathbf{Z}_+$  with mean  $\lambda$ , then

$$\begin{aligned} D(T_{1/n}(f^{*n})) &\leq \{n\lambda\} D(bi(\lfloor n\lambda \rfloor + 1, 1/n) | po(\lambda)) \\ &\quad + (1 - \{n\lambda\}) D(bi(\lfloor n\lambda \rfloor, 1/n) | po(\lambda)) \end{aligned} \quad (20)$$

where  $\{x\}$  and  $\lfloor x \rfloor$  denote the fractional and integer parts of  $x$ , respectively.

Theorem 8 and (19) easily yield

$$D(T_{1/n}(f^{*n})) = O(n^{-2}), \quad n \rightarrow \infty,$$

as long as  $f$  is ULC. To prove Theorem 8, we again adopt the strategy of Section VI. Proposition 8 is a variant of Lemma 4.

**Proposition 8:** Let  $f$  and  $g$  be pmfs on  $\mathbf{Z}_+$  such that  $f \leq_{cx} g$  and  $g$  is ULC. Then

$$D(f) \geq D(g) + D(f|g).$$

We also have the following result, which is easily deduced from Theorem 3.A.13 of Shaked and Shanthikumar [31] (see also [39], Lemma 2). Plainly, it says that the convex order  $\leq_{cx}$  is preserved under thinning.

**Proposition 9:** If  $f$  and  $g$  are pmfs on  $\mathbf{Z}_+$  such that  $f \leq_{cx} g$ , then  $T_\alpha f \leq_{cx} T_\alpha g$ ,  $\alpha \in (0, 1)$ .

*Proof of Theorem 8:* Let  $g$  be the two-point pmf that assigns probability  $\{n\lambda\}$  to  $\lfloor n\lambda \rfloor + 1$  and the remaining probability to  $\lfloor n\lambda \rfloor$ . The mean of  $g$  is  $n\lambda$ . The relation  $g \leq_{cx} f^{*n}$  is intuitive and easily proven. By Proposition 9,  $T_{1/n}g \leq_{cx} T_{1/n}(f^{*n})$ . Since  $f$  is ULC, so is  $T_{1/n}(f^{*n})$ . By Proposition 8,  $D(T_{1/n}(f^{*n})) \leq D(T_{1/n}g)$ . However  $T_{1/n}g$  is a mixture of two binomials:

$$T_{1/n}g = \{n\lambda\} bi(\lfloor n\lambda \rfloor + 1, 1/n) + (1 - \{n\lambda\}) bi(\lfloor n\lambda \rfloor, 1/n).$$

Thus (20) holds by the convexity of the relative entropy. ■

It would be interesting to see the ULC assumption relaxed. Theorem 9 shows that the  $O(n^{-2})$  rate holds under a finite support assumption. Note that, in the CLT case, an  $O(n^{-1})$  rate of convergence for the relative entropy can be obtained under a ‘‘spectral gap’’ assumption ([1], [21]); possibly a similar assumption suffices in our case. Under the finite support assumption, however, the proof of Theorem 9 is elementary, although it does use a nontrivial subadditivity property of the scaled Fisher information ([24], [28]).

**Theorem 9:** Suppose  $f$  is a pmf on  $\mathbf{Z}_+$  with finite support and denote the mean and variance of  $f$  by  $\lambda$  and  $\sigma^2$  respectively. Then

$$D(T_{1/n}(f^{*n})) = O(n^{-2}), \quad n \rightarrow \infty. \quad (21)$$

If  $\lambda = \sigma^2$  in addition, then the right hand side of (21) can be replaced by  $O(n^{-3})$ .

*Proof:* Let us assume  $\lambda > 0$  to eliminate the trivial case. For a pmf  $g$  on  $\mathbf{Z}_+$  with mean  $\mu > 0$ , define  $K(g) = \mu\chi^2(S(g), g)$  as in (9). Madiman et al. ([28], Theorem III) show that  $K(g^{*n})$  decreases in  $n$ . In particular, letting  $g = T_{1/n}(f)$ , and noting (9) and (2), we obtain

$$D(T_{1/n}(f^{*n})) \leq K(T_{1/n}(f^{*n})) \leq K(T_{1/n}(f)).$$

Thus, to prove (21), we only need  $K(T_{1/n}(f)) = O(n^{-2})$ . By the definition of  $K(\cdot)$  and (6), this is equivalent to

$$\chi^2(T_{1/n}(S(f)), T_{1/n}(f)) = O(n^{-1}). \quad (22)$$

Let  $k$  be the largest integer such that  $f_k \neq 0$ . We have, for  $0 \leq i \leq k$ ,

$$\begin{aligned} (T_{1/n}(f))_i &= \sum_{j=i}^k f_j bi(i, j, 1/n) \\ &= n^{-i} \sum_{j=i}^k \binom{j}{i} f_j + O(n^{-i-1}). \end{aligned}$$

A similar expression holds for  $T_{1/n}(S(f))$ . By direct calculation, each term in the sum

$$\sum_{i=0}^k \frac{((T_{1/n}(S(f)))_i - (T_{1/n}(f))_i)^2}{(T_{1/n}(f))_i} \quad (23)$$

is  $O(n^{-1})$ , and (22) holds. If  $\lambda = \sigma^2$ , then each term in (23) is  $O(n^{-2})$ , thus proving the remaining claim. ■

Theorems 8 and 9 imply a corresponding rate of convergence for the total variation distance, which is defined as  $V(g, \tilde{g}) = \sum_i |g_i - \tilde{g}_i|$  for any pmfs  $g$  and  $\tilde{g}$ . The total variation is related to the relative entropy via Pinsker’s inequality

$V^2(g, \tilde{g}) \leq 2D(g|\tilde{g})$ . Hence, if  $f$  is either ULC or has finite support, then

$$V(T_{1/n}(f^{*n}), p\circ(\lambda)) = O(n^{-1}).$$

An explicit upper bound, possibly via the Stein-Chen method, is of course desirable.

### VIII. SUMMARY AND POSSIBLE EXTENSIONS

We have extended the monotonicity of entropy in the central limit theorem to a version of the law of small numbers considered by [15], [16], [17]. This involves the thinning operation (the discrete analogue of scaling), and a Poisson limit (the discrete counterpart of the normal). For a pmf  $f$  on  $\mathbf{Z}_+$  with mean  $\lambda$ , we show that the relative entropy  $D(T_{1/n}(f^{*n})|p\circ(\lambda))$  decreases monotonically in  $n$  (Theorem 2), and, if  $f$  is ultra-log-concave, the entropy  $H(T_{1/n}(f^{*n}))$  increases in  $n$  (Theorem 3). In the process of establishing Theorem 2, inequalities are obtained for the relative entropy under thinning and convolution, and connections are made with logarithmic Sobolev inequalities and with the recent results of Kontoyiannis et al. [24] and Madiman et al. [28]. Theorem 3, in contrast, is established by comparing pmfs with respect to the convex order, an idea that dates back to Karlin and Rinott [23].

This work is arguably more qualitative than quantitative, given its focus on monotonicity. When bounds are occasionally obtained, in Proposition 2 for example, we do not claim that they are always sharp. Among the large literature on Poisson approximation bounds (e.g., Barbour et al. [3]), the use of information theoretic ideas is a relatively new development ([24], [28]). We have, however, obtained an upper bound and identified an  $O(n^{-2})$  rate for the relative entropy under certain simple conditions. Such results complement those of [15], [16], [17].

The analogy with the CLT leads to further questions. As pointed out by a referee, monotonicity of entropy in the CLT follows from more general inequalities for sums of independent random variables ([2], [27]). In contrast, our Theorem 3 establishes monotonicity of entropy in its own right. This suggests room for future work, which may lead to a discrete version of the entropy power inequality (EPI). Certain simple formulations of the EPI do not hold in the discrete setting; see [42] for recent developments.

We may also consider extending our monotonicity results to compound Poisson limit theorems. Recently, Johnson et al. [22] (see also [39]) have shown that compound Poisson distributions admit a maximum entropy characterization similar to that of the Poisson. Such results suggest the possibility of compound Poisson limit theorems with the same appealing ‘‘entropy increasing to the maximum’’ interpretation.

Finally, on a more technical note, we point out a possible refinement of Theorem 2. This is analogous to the results of Yu [40], who noted that relative entropy is *completely monotonic* in the CLT for certain distribution families. (A function is completely monotonic if its derivatives of all orders exist and alternate in sign; the definition is similar for discrete sequences; see Feller [12] for the precise statements.)

**Theorem 10 ([40]):** Let  $X_i$ ,  $i = 1, 2, \dots$ , be i.i.d. random variables with distribution  $F$ , mean  $\mu$ , and variance  $\sigma^2 \in (0, \infty)$ . Then  $D\left(\sum_{i=1}^n (X_i - \mu)/\sqrt{n\sigma^2} | N(0, 1)\right)$  is a completely monotonic function of  $n$  if  $F$  is either a gamma distribution or an inverse Gaussian distribution.

Part of the reason that the gamma and inverse Gaussian distributions are considered is that they are analytically tractable. The result may conceivably hold for a wide class of distributions. We conclude with a discrete analogue based on numerical evidence.

**Conjecture 1:** Let  $\lambda > 0$ . Then

- $D(\text{bi}(n, \lambda/n))$  is completely monotonic in  $n$  ( $n \geq \lambda$ );
- $D(\text{nb}(n, n/(\lambda + n)))$  is completely monotonic in  $n$  ( $n > 0$ ).

We again expect similar results for other pmfs, but are unable to prove even those for the binomial and the negative binomial.

### APPENDIX

#### PROOF OF PROPOSITION 2

Let  $g = f^{*n}$ . Following the proof of Proposition 8 in [15], we use convexity to obtain

$$\begin{aligned} D(T_{1/n}(g)) &= D\left(\sum_{k=0}^{\infty} g_k \text{bi}(k, 1/n)\right) \\ &\leq \sum_{k=0}^{\infty} g_k D(\text{bi}(k, 1/n)|p\circ(\lambda)). \end{aligned}$$

However,

$$\begin{aligned} D(\text{bi}(k, p)|p\circ(\lambda)) &= D(\text{bi}(k, p)) + D(p\circ(kp)|p\circ(\lambda)) \\ &\leq kp^2 + kp \log \frac{kp}{\lambda} - kp + \lambda \end{aligned}$$

where the simple bound  $D(\text{bi}(k, p)) \leq kp^2$  (see [14] for its proof) is used in the inequality. Thus

$$\begin{aligned} D(T_{1/n}(g)) &\leq \sum_{k=0}^{\infty} g_k \left[ \frac{k}{n^2} + \frac{k}{n} \log \frac{k}{n\lambda} - \frac{k}{n} + \lambda \right] \\ &= \frac{\lambda}{n} + E\bar{X}_n \log \frac{\bar{X}_n}{\lambda} \end{aligned}$$

as required.

#### PROOF OF PROPOSITION 6

Let us recall a well-known characterization of the convex order (see [31], Theorem 3.A.1, for example).

**Proposition 10:** Let  $X$  and  $Y$  be random variables on  $\mathbf{Z}_+$  such that  $EX = EY < \infty$ . Then  $X \leq_{cx} Y$  if and only if

$$E \max\{X - k, 0\} \leq E \max\{Y - k, 0\}, \quad k \geq 0,$$

or, equivalently,

$$\sum_{i \geq k} \Pr(X \geq i) \leq \sum_{i \geq k} \Pr(Y \geq i), \quad k \geq 0.$$

**Proposition 11:** Fix  $p \in (0, 1)$ , and let  $Y_1$  and  $Y_2$  be i.i.d. random variables on  $\mathbf{Z}_+$  with an ultra-log-concave pmf  $f$ . Let

$Z_1, Z_2, \tilde{Z}_1$  and  $\tilde{Z}_2$  be independent conditional on  $Y_1$  and  $Y_2$  and satisfy

$$\begin{aligned} Z_1|Y_1 &\sim \text{Bi}(Y_1, p + \delta), & Z_2|Y_2 &\sim \text{Bi}(Y_2, p - \delta), \\ \tilde{Z}_1|Y_1 &\sim \text{Bi}(Y_1, p + \tilde{\delta}), & \tilde{Z}_2|Y_2 &\sim \text{Bi}(Y_2, p - \tilde{\delta}). \end{aligned}$$

If  $\delta > \tilde{\delta} \geq 0$ , then  $Z_1 + Z_2 \leq_{cx} \tilde{Z}_1 + \tilde{Z}_2$ .

*Proof:* We show that, for each  $k \geq 0$ ,  $\sum_{i \geq k} \Pr(Z_1 + Z_2 \geq i)$  is a decreasing function of  $\delta$  as long as  $0 \leq \delta \leq \min\{p, 1 - p\}$ . The claim then follows from Proposition 10 (the assumptions imply  $E(Z_1 + Z_2) = E(\tilde{Z}_1 + \tilde{Z}_2) < \infty$ ). To simplify the notation, in what follows the limits of summation, if not spelled out, are from  $-\infty$  to  $\infty$ ; also  $f_i \equiv 0$  if  $i < 0$ . Denoting  $B(i; n, p) = \sum_{j \geq i} bi(j; n, p)$ , and letting  $h(\delta) = \sum_{i \geq k} \Pr(Z_1 + Z_2 \geq i)$ , we have

$$\begin{aligned} h(\delta) &= \sum_{i \geq k} \sum_j \Pr(Z_1 \geq j) \Pr(Z_2 = i - j) \\ &= \sum_j \Pr(Z_1 \geq j) \Pr(Z_2 \geq k - j) \\ &= \sum_j \left[ \sum_{s \geq 0} f_s B(j; s, p + \delta) \right] \left[ \sum_{s \geq 0} f_s B(k - j; s, p - \delta) \right] \\ &= \sum_{s, t \geq 0} f_s f_t v(s, t, \delta) \end{aligned} \quad (24)$$

where

$$v(s, t, \delta) = \sum_j B(j; s, p + \delta) B(k - j; t, p - \delta).$$

Using the simple identity

$$\frac{dB(i; n, p)}{dp} = n[bi(i - 1; n - 1, p)]$$

we get

$$\frac{dv(s, t, \delta)}{d\delta} = su(s, t, \delta) - tu(t, s, -\delta)$$

where

$$\begin{aligned} u(s, t, \delta) &= \sum_j bi(j - 1; s - 1, p + \delta) B(k - j; t, p - \delta) \\ &= \sum_j bi(k - j - 1; s - 1, p + \delta) B(j; t, p - \delta). \end{aligned}$$

The quantity  $u(s, t, \delta)$  has the following interpretation. If we let  $V_1 \sim \text{Bi}(s - 1, p + \delta)$  and  $V_2 \sim \text{Bi}(t, p - \delta)$  independently, then  $u(s, t, \delta) = \Pr(V_1 + V_2 \geq k - 1)$ . Clearly

$$u(s, t, \delta) = u(t + 1, s - 1, -\delta). \quad (25)$$

Hence, we may take the derivative under the summation in (24) (by dominated convergence), and then apply (25) to obtain

$$\begin{aligned} h'(\delta) &= \sum_{s, t \geq 0} s f_s f_t u(s, t, \delta) - \sum_{s, t \geq 0} t f_s f_t u(t, s, -\delta) \\ &= \sum_{s \geq 1, t \geq 0} s f_s f_t u(s, t, \delta) \\ &\quad - \sum_{s \geq 1, t \geq 0} (t + 1) f_{s-1} f_{t+1} u(t + 1, s - 1, -\delta) \\ &= \sum_{s \geq 1, t \geq 0} [s f_s f_t - (t + 1) f_{s-1} f_{t+1}] u(s, t, \delta). \end{aligned} \quad (26)$$

By splitting the sum (26) into two regions  $1 \leq s \leq t$  and  $0 \leq t \leq s - 2$ , and by a change of variables  $s \rightarrow t + 1$  and  $t \rightarrow s - 1$  in the latter region, we get

$$\begin{aligned} h'(\delta) &= \sum_{1 \leq s \leq t} [s f_s f_t - (t + 1) f_{s-1} f_{t+1}] \\ &\quad \times [u(s, t, \delta) - u(t + 1, s - 1, \delta)] \\ &= \sum_{1 \leq s \leq t} [s f_s f_t - (t + 1) f_{s-1} f_{t+1}] \\ &\quad \times [u(s, t, \delta) - u(s, t, -\delta)] \end{aligned} \quad (27)$$

where (27) follows from (25). Because  $f$  is ULC, if  $s \leq t$ , then

$$s f_s f_t \geq (t + 1) f_{s-1} f_{t+1}. \quad (28)$$

We can also show ( $s \leq t$ )

$$u(s, t, \delta) \leq u(s, t, -\delta) \quad (29)$$

as follows. Let  $W_1, W_2, W_3, W_4$  be independent random variables such that

$$\begin{aligned} W_1 &\sim \text{Bi}(s - 1, p + \delta), & W_2 &\sim \text{Bi}(s - 1, p - \delta), \\ W_3 &\sim \text{Bi}(t - s + 1, p + \delta), & W_4 &\sim \text{Bi}(t - s + 1, p - \delta). \end{aligned}$$

Then

$$\begin{aligned} u(s, t, \delta) &= \Pr(W_1 + W_2 + W_4 \geq k - 1); \\ u(s, t, -\delta) &= \Pr(W_1 + W_2 + W_3 \geq k - 1). \end{aligned}$$

Since  $\delta \geq 0$ , we have  $W_4 \leq_{st} W_3$ , which yields  $W_1 + W_2 + W_4 \leq_{st} W_1 + W_2 + W_3$ , and  $u(s, t, \delta) \leq u(s, t, -\delta)$  by the definition of  $\leq_{st}$ . Now (27), (28) and (29) give

$$h'(\delta) \leq 0$$

i.e.,  $h(\delta)$  decreases in  $\delta$ . ■

*Proof of Proposition 6:* Given the basic properties of majorization, we only need to prove that  $E\phi(\sum_{i=1}^n Z_i)$  is Schur concave as a function of  $(p_1, p_2)$  holding  $p_3, \dots, p_n$  fixed. Define  $\psi(z) = E\phi(z + \sum_{i=3}^n Z_i)$ . Since  $\phi$  is convex, so is  $\psi$ . (We may assume that  $\psi$  is finite as the general case can be handled by a standard limiting argument.) Proposition 11, however, shows precisely that  $E\psi(Z_1 + Z_2) = E\phi(\sum_{i=1}^n Z_i)$  is Schur-concave in  $(p_1, p_2)$ . ■

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