
sensitivity = P(T+ | BC) = 99%

specificity = P(T- | BC) = 99.5%

We also know the prevalence is 1.2%, i.e., P(BC) = 1.2%.

Q: What is the probability that a woman has BC if the test result is positive?

Solution. The question asks P(BC | T+). By the Bayes' rule,

\[
P(BC | T^+) = \frac{P(T^+ | BC) P(BC)}{P(T^+ | BC) P(BC) + P(T^+ | \overline{BC}) P(\overline{BC})} = \frac{0.99 \times 0.012}{0.99 \times 0.012 + (1-0.99) \times (1-0.012)} = 70.6\%\]

Independent Events

We say events A and B are independent if knowing that B occurred doesn't change the probability of A (or vice versa), i.e., \( P(\overline{A}) = P(A) \) or \( P(B | A) = P(B) \) \( \Rightarrow P(A \cap B) = P(A) P(B) \)

Def. We say A and B are independent if \( P(AB) = P(A) P(B) \). The formal definition of independence.

2.2.1 Notation \( A,B \)

Thm 2.2.2 \( A \cap B \Rightarrow A \cap B \Rightarrow A \cap B \Rightarrow A \cap B \)

Proof. We only provide the proof of \( A \cap B \Rightarrow A \cap B \). The rest of proof is similar.

Recall that \( A \cap B \) and \( A \cap B \) form a partition of \( B \). So

\[
P(\overline{A \cap B}) + P(A \cap B) = P(B) \Rightarrow P(\overline{A \cap B}) = P(C) - P(A \cap B)\]

\[
A \cap B \Rightarrow P(ABC) = P(A) P(B) \Rightarrow P(C \cap AB) = P(B) - P(C) P(B) = P(B)(1 - P(A)) = P(B) P(\overline{A})
\]

\[\Rightarrow A \cap B \]

More than two events?

Pairwise independence. Events \( A_1, \ldots, A_k \) are pairwise independent if for every pair \( A_i \) and \( A_j \)\( (i \neq j) \), \( P(A_i \cap A_j) = P(A_i) P(A_j) \)

E.g. simple sample space \( \Omega = \{1,2,3,4\} \)

\[\begin{array}{cccc}
A_1 &=& 1 & \Rightarrow P(A_1) = \frac{1}{4} \\
A_2 &=& 2 & \Rightarrow P(A_2) = \frac{1}{4} \\
A_3 &=& 3 & \Rightarrow P(A_3) = \frac{1}{4} \\
A_4 &=& 4 & \Rightarrow P(A_4) = \frac{1}{4}
\end{array}\]

\( A_1 \cup A_2 \cup A_3 \cup A_4 \)

Similarity. \( A_1, A_2, \ldots, A_k \) are independent.

Independent (Mutually independent). Events \( A_1, A_2, \ldots, A_k \) are mutually independent if

\[P(A_1 \cap A_2 \cap \ldots \cap A_m) = P(A_1) P(A_2) \cdots P(A_m)\]

for every subset \( \{A_1, A_2, \ldots, A_m\} \) of \( m \) of these events \( (m = 2, 3, \ldots, k) \)

Pairwise independent \( \Rightarrow \) Mutually independent. In the e.g. above,

\[P(A_1 \cap A_2 \cap A_3) = \frac{1}{6} = P(A_1) P(A_2) P(A_3)\]

\[\Rightarrow\]
e.g. Toss a fair coin three times

A = H on 1st toss = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}

B = H on 2nd toss = \{HHH, HHT, THT, TTH\}

C = H on 3rd toss = \{HHH, HHT, THT, TTH\}

\[ P(A \cap B) = P(HH) = \frac{1}{4} = P(A)P(B) \Rightarrow A,B \text{ are independent} \]

\[ P(A \cap C) = P(HH) = \frac{1}{4} = P(A)P(C) \Rightarrow A,C \text{ are independent} \]

\[ P(B \cap C) = P(HH) = \frac{1}{4} = P(B)P(C) \Rightarrow B,C \text{ are independent} \]

\[ \Rightarrow A,B,C \text{ are independent (mutually independent)} \]

When there are \( >2 \) events, "independent" is a sloppy way for "mutually independent".

**Sampling with vs without replacement**

52 card deck - 12 face cards

Choose 5 cards at random, one at a time

Let \( A_i \) = the \( i \)th card chosen is a face card \( (i=1, 2, \ldots, 5) \)

0. Sampling with replacement

\( A_1, A_2, \ldots, A_5 \) are mutually independent

2. Sampling without replacement

\[ P(A_1)P(A_2|A_1)P(A_3|A_1A_2)P(A_4|A_1A_2A_3)P(A_5|A_1A_2A_3A_4) \]

\[ = \frac{12}{52} \times \frac{11}{51} \times \frac{10}{50} \times \frac{9}{49} \times \frac{8}{48} = \left( \frac{12}{52} \right)^5 \]

**Independent vs Disjoint**

They are not the same. They cannot happen at the same time. Assume \( P(A) > 0, P(B) > 0 \)

If \( A \cap B = \emptyset \), then \( A \) and \( B \) are not independent.

On the other hand, if \( A \cup B \), then \( A \) and \( B \) are not disjoint, i.e., \( A \cap B \neq \emptyset \).

Eg. Machine 1 and Machine 2 are operated independently of each other.

\( A = \text{Machine 1 fails within an 8-hour period}, \ P(A) = \frac{1}{2} \)

\( B = \text{Machine 2 fails}, \ P(B) = \frac{1}{4} \)

What is the probability that at least one machine fails?

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

Because \( A \cap B \), \( P(A \cap B) = P(A)P(B) = \frac{1}{8} \)
Def 2.2.3 \( A_1, \ldots, A_k \) are conditional independent given \( B \) if for every subset \( A_{i_1}, \ldots, A_{i_m}(m \leq k) \):
\[
P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m} | B) = \prod_{j=1}^{m} P(A_{i_j} | B)
\]

- e.g. two coins — one is fair, one is biased
  - Fair coin \( \Rightarrow P(H) = \frac{1}{2} \)
  - Biased coin \( \Rightarrow P(H) = P > \frac{1}{2} \)
  - Pick a coin at random and toss it twice independently
    \[
    \begin{align*}
    A_1 &= \text{head on 1st toss} \quad P(B) = P(B|C) = \frac{1}{2} \\
    A_2 &= \text{head on 2nd toss} \quad P(A_1 | B) = P(A_1 | B) = P \\
    B &= \text{coin is biased} \quad P(A_1 | B^c) = P(A_1 | B^c) = \frac{1}{2}
    \end{align*}
    \]

  So \( A_1 \) and \( A_2 \) are independent if the biased coin is picked

\[
\begin{align*}
(A, A_1) &\perp B \\
\frac{P(A_1 | A_2 | B)}{P(A_1 | B)} &= \frac{P(A_1 | B)}{P(A_1 | B)} = \frac{P^2}{P^2} = 1
\end{align*}
\]

\[
P(A_1 A_2 | B) = P(A_1 | B) P(A_2 | B) = P^2
\]

\[
P(A_1 A_2 | B^c) = P(A_1 | B^c) P(A_2 | B^c) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}
\]

**Question:** Are \( A_1 \) and \( A_2 \) independent of each other?

\[
P(A_1) = P(A_1 | B) + P(A_1 | B^c) = P(B) P(A_1) + P(B^c) P(A_1 | B^c) = \frac{1 + 2P}{4}
\]

**Similarly,**

\[
P(A_2) = \frac{1 + 2P}{4}
\]

\[
P(A_1 A_2) = P(A_1 | A_2 | B) P(B) + P(A_1 | A_2 | B^c) P(B^c)
\]

\[
\begin{align*}
&= P(A_1 | B) P(A_2 | B) P(B) + P(A_1 | B^c) P(A_2 | B^c) P(B^c) \\
&= P \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1 + 2P}{8}
\end{align*}
\]

But \( P(A_1) P(A_2) = \left( \frac{1 + 2P}{4} \right)^2 P(A_1) P(A_2) \)

So \( A_1 \perp A_2 \).