e.g. (continued) $X$ = # of 0−donors in a sample of 3 donors

$\text{Support } X = \{0, 1, 2, 3\}$

$\text{pmf: } f_X(x) = \binom{3}{x} 0.01^x (1-0.01)^{3-x}, \quad x \in X$

For $x < 0$, $F(x) = 0$

$0 \leq x < 1$, $F(x) = P(X=0) = 0.0044$

$1 \leq x < 2$, $F(x) = P(X=0) + P(X=1)$

$2 \leq x < 3$, $F(x) = P(X=0) + P(X=1) + P(X=2)$

$3 \leq x$, $F(x) = 1$

For a discrete r.v., the cdf is a step function.

E.g. Buses at a bus stop arrive at 6 pm and 7 pm. You arrive at the stop at random between 6 pm and 7 pm.

Let $X$ denote the number of minutes you have to wait for a bus.

$\text{Support } X = (0, 60)$

$X$ is a continuous r.v.

Because the arrival time is random between 6 pm and 7 pm, the pdf should be flat on $(0, 60)$.

Need to determine $a$.

Since $P(0 \leq X < 60) = 1$

$1 = \int_{0}^{60} f(x) \, dx = \int_{0}^{60} a \, dx \implies a = \frac{1}{60}$

$\text{pdf: } f_X(x) = \frac{1}{60}, \quad 0 < x < 60 \quad \text{or} \quad f_X(x) = 0 \text{ elsewhere}$

$\text{cdf: } F_X(x) = \int_{0}^{x} f_X(t) \, dt = \int_{0}^{x} \frac{1}{60} \, dt = \frac{x}{60}$

$0 < x < 60$, $F(x) = \frac{x}{60}$

$x \geq 60$, $F(x) = 1$
What is the probability that I have to wait at least 40 mins?

\[ P(X > 40) = \int_{40}^{60} f(x) \, dx = \frac{60 - 40}{60} = \frac{1}{3} \]

What is the probability that the waiting time is less than 5 mins?

\[ P(X < 5) = \int_{0}^{5} f(x) \, dx = \frac{5}{60} = \frac{1}{12} \]

**Fact 1:** Continuous distributions assign probability to individual values.

If \( X \) is a continuous r.v., \( P(X = x) = 0 \). As a result,

\[ P(a < X < b) = P(a \leq X \leq b) = P(a < X < b) = P(a < X \leq b) \]

**Why?** For every \( \varepsilon > 0 \) and a s.t. \( f(a) > 0 \)

\[ P(a - \varepsilon < X < a + \varepsilon) \approx 2\varepsilon f(a) > 0 \]

\[ \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \]

**Fact 2:** For a continuous r.v., its cdf is continuous.

**A general Uniform distribution**

If \( X \) has a Uniform distribution over the interval \((a, b)\), then its pdf is

\[ f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{elsewhere} \end{cases} \]

and cdf is

\[ F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \geq b \end{cases} \]

**Note:** Density ≠ Probability

A density can be greater than 1, but probability cannot.

\( X \sim \text{Unif}(0, \frac{1}{2}) \)

\[ f(x) = \begin{cases} 2 & 0 < x < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases} \]
Properties of pmf, pdf, and cdf

**Discrete r.v. X:** If \( f(x) \) is a pmf, then
1. \( f(x) \geq 0, \forall x \in \mathbb{R} \)
2. \( \sum_{x} f(x) = 1 \)

**Continuous r.v. X:** If \( f(x) \) is a pdf, then
1. \( f(x) \geq 0, \forall x \in \mathbb{R} \)
2. \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \)
3. \( P(a \leq X < b) = \int_{a}^{b} f(x) \, dx \) for \( a < b \)

For any type of r.v. X, the cdf is defined as \( F(x) = P(X \leq x) \). It satisfies,
1. Nondecreasing: if \( x_1 < x_2 \), then \( F(x_1) \leq F(x_2) \)
2. \( \lim_{x \to -\infty} F(x) = 0 \), \( \lim_{x \to \infty} F(x) = 1 \)
3. Right continuous: \( \lim_{x \to a^+} F(x) = F(a) \)

**Common Distributions**

We have seen two distributions: Binomial(\( n, p \)) \( X = 0, 1, 2, \ldots, n \) and Uniform(\( a, b \)) \( X = (a, b) \)

We want to verify that
- \( \sum_{x} f(x) = 1 \) or \( \int f(x) \, dx = 1 \)
- Binomial Theory
  \[ f(x) = \binom{n}{x} p^x (1-p)^{n-x} \]
  \[ F(x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} \]
- Uniform Theory
  \[ f(x) = \frac{1}{b-a} \]
  \[ F(x) = \frac{x-a}{b-a} \]

 Binomial Theory
\[ f(x) = \binom{n}{x} p^x (1-p)^{n-x} \]
\[ F(x) = \sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k} \]

 Uniform Theory
\[ f(x) = \frac{1}{b-a} \]
\[ F(x) = \frac{x-a}{b-a} \]
Bernoulli$(p)$, $0 < p < 1$

It is the same as Binomial$(1, p)$. Bernoulli is a special Binomial.

Let $X \sim \text{Bernoulli}(p)$, $X = 50.15$

$$f(x) = \binom{x}{r} p^x (1-p)^{1-x} = p^x (1-p)^{1-x} = \begin{cases} P & \text{if } x = 1 \\ 1-P & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

So $X$ is binary. Only takes two possible values; 0, 1.

$$\sum_{x=0}^{1} f(x) = f(0) + f(1) = p + (1-p) = 1$$

eg. Flip a fair coin. Let $X = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$

Then $X \sim \text{Bernoulli}(p)$.

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{Bernoulli}(p)$

"iid": independent and identically distributed

note: if $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \text{distribution}$, we usually say $X_1, X_2, \ldots, X_n$ is a random sample from the distribution.

Then $Y = X_1 + X_2 + \ldots + X_n \sim \text{Binomial}(n, p)$

The proof will be given later using moment generating function (mgf).

The interpretation of the result: $Y$ is the number of successes in $n$ independent Bernoulli trials, each with a success probability of $p$.

eg. There are $n$ independent fair coins

$X_i$: the number of heads when flip the $i$th coin. $X_i = 0$ or 1. $X_i \overset{iid}{\sim} \text{Bernoulli}(1)$

$Y$: the number of heads when flip $n$ fair coins. $Y \overset{\text{Binomial}}{\sim} (n, p)$

pmf: $f_1(x) = p^x (1-p)^{1-x}$, $x = 0$ or 1

pmf: $f_2(y) = \binom{n}{y} p^y (1-p)^{n-y}$

Discrete Uniform

pmf: $f(x) = P(X=x) = \begin{cases} \frac{1}{b-a+1} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

$$\sum_{x=a}^{b} f(x) = \frac{1}{b-a+1} + \frac{1}{b-a+1} + \ldots + \frac{1}{b-a+1} = \frac{b-a+1}{b-a+1} = 1$$

e.g. roll a well-balanced dice.

Let $X$: face value. $f(x) = P(X=x) = \begin{cases} \frac{1}{6} & x = 1, 2, \ldots, 6 \\ 0 & \text{otherwise} \end{cases}$