Classical Probability

Suppose the outcomes of \( \Omega \) all have equal probability. Let \( A \) be an event. Then
\[
P(A) = \frac{\# \text{ ways } A \text{ occurs}}{\# \text{ outcomes in } \Omega}
\]

\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}, \quad \text{is called "binomial coefficients" because}
\]
\[
(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}
\]

Remarks:
\[
\begin{align*}
n! &= n \times (n-1) \times \cdots \times 1 \\
0! &= 1 \\
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\end{align*}
\]

Example. A person can choose six numbers from 1, 2, ..., 44.

The winning number is then decided by randomly selecting six numbers from the 44.

Q: how many ways of picking 6 numbers?

Before answering the question, we need to know
1) whether replacement is allowed?
2) does order matter?

A. Ordered & without replacement
\[
\frac{44 \times 43 \times 42 \times 41 \times 40 \times 39}{6!} = \frac{44!}{38!} = \frac{44!}{(44-6)!} \approx 5 \times 10^9
\]

B. Ordered & with replacement
\[
44 \times 44 \times 44 \times 44 \times 44 \times 44 = 44^6 \approx 7 \times 10^9
\]

C. Unordered & without replacement
\[
\frac{44 \times 43 \times \cdots \times 39}{6!} = \frac{44!}{6!} \left(\frac{44}{6}\right)! = \frac{44!}{6!(44-6)!} = 6! \left(\frac{44}{6}\right)! = (\frac{44}{6}) \approx 7 \times 10^6
\]

D. Unordered & with replacement

this situation is more complicated than A, B, C.

we will show two methods to solve it.
method 1
under the assumption of "unordered & with replacement," each way of
choosing six numbers corresponds to a way of putting six balls (not labeled)
into 44 numbered bins. For example,

$$\{1, 3, 3, 4, 41, 42\} \leftrightarrow \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & \cdots \\
6 & 41 & 42 & 43 & 44
\end{array}$$

So the answer = # ways of arranging "|" and "_" (6 balls)
(45 walls)
The two outermost positions are fixed, i.e., they must be walls.
So they are \((45+6-2) = 49\) positions left, out of which
six balls will be placed. There are
\[
\binom{49}{6} = \frac{49!}{6!43!} \approx 1.4 \times 10^7 \text{ ways}
\]

method 2

step 1. determine the how to choose \(k\) distinct numbers from 1...44,
where \(k = 1, 2, \ldots, 6\); \(\binom{44}{k}\)

step 2. For each \(k\), find the number of ways of making 6 numbers
using the \(k\) distinct numbers.
this problem is equivalent to find how many solutions for
\(x_1 + x_2 + \cdots + x_k = 6\), where \(x_i \geq 1\) for \(i = 1, \ldots, k\)
\[
\downarrow
\]
\(x_1^* + x_2^* + \cdots + x_k^* = 6-k\), where \(x_i^* \geq 0\) for \(i = 1, \ldots, k\)
arrange \(k-1\) inner walls, 6-k balls: \(\binom{k-1+6-k}{k-1} = \binom{5}{k-1}\)

So the total number of ways is
\[
\binom{44}{1} + \binom{44}{2} + \binom{44}{3} + \binom{44}{4} + \binom{44}{5} + \binom{44}{6} + \binom{44}{7} + \binom{44}{8} \approx 1.4 \times 10^7
\]

Summary: number of possible arrangements of size \(r\) from \(n\) objects

<table>
<thead>
<tr>
<th>ordered</th>
<th>unordered</th>
</tr>
</thead>
<tbody>
<tr>
<td>without replacement</td>
<td>(\frac{n^r}{(nr)!})</td>
</tr>
<tr>
<td>with replacement</td>
<td>(\binom{n}{r})</td>
</tr>
</tbody>
</table>
Birthday Problem

1. Suppose that a room contains \( n \) people. What is the probability that at least two of them have a common birthday?

\[
P(\text{at least two people have the same birthday}) = 1 - P(\text{there are } n \text{ unique birthdays})
\]

\[
= 1 - P(\text{person 2 cannot take person 1's birthday}) \cdot P(\text{person 3 cannot take persons 1-2's birthdays}) \cdots
\]

\[
= 1 - \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+1}{365}
\]

\[
= 1 - \frac{365 \cdot 364 \cdots (365-n+1)}{365^n}
\]

\[
\begin{array}{c|c}
 n & P \\hline
 4 & 0.016 \\
 10 & 0.284 \\
 23 & 0.507 \\
 32 & 0.673 \\
 40 & 0.791 \\
 50 & 0.988 \\
 98 & 0.9999994 \\
\end{array}
\]

R code

\[
n=16; \text{ choose}(365,n) \times \text{factorial}(n) / 365^n
\]

2. Suppose you ask \( n \) people. What is the probability of finding at least one person with the same birthday of yours?

\[
P(\text{finding at least one}) = 1 - P(\text{didn't find any person})
\]

\[
= 1 - (1 - \frac{1}{365})^n
\]

\[
= 1 - \left(\frac{364}{365}\right)^n
\]