1. Let $X_{n \times p}$ be a design matrix. Suppose that $\text{rank}(X) = r < p$. Consider the normal equation

$$X^T X \beta = X^T Y$$

(a) In class we showed that $\hat{\beta} = (X^T X)^{-1} X^T Y$ is a solution to the normal equation, where $(X^T X)^{-1}$ is a generalized inverse. Show that

$$\tilde{\beta} = \hat{\beta} + (I - (X^T X)^{-1}(X^T X))z$$

where $z$ is an arbitrary $p \times 1$ vector, is also a solution to the normal equation.

(b) Without loss of generality, assume $X = (X_1, X_2)$ where $\text{rank}(X_1) = r$. Prove the following.

i. There exists a $r \times (p - r)$ matrix $C$ such that $X_2 = X_1 C$.

ii. $\hat{\beta} = \left( (X_1^T X_1)^{-1} X_1^T Y \right) 0$ is a solution to the normal equation.

2. Consider the model

$$Y_{ij} = \alpha_i + \beta_j + \epsilon_{ij}$$

where $i = 1, \ldots, a ; j = 1, \ldots, b$ and $\epsilon_{ij} \sim (0, \sigma^2)$. Derive the necessary condition that I gave in class for $\sum c_i \alpha_i + \sum d_j \beta_j$ to be estimable.

3. Consider a full rank $n \times p$ design matrix $X = (1, x_1, \cdots, x_{p-1})$. In design matrix $X$, the first column is a vector of 1’s, and $x_i$ is the $(i+1)th$ column, for $i = 1, \cdots, p-1$. Let $Z = (1, c_1 x_1, \cdots, c_{p-1} x_{p-1})$ be a linear transformation of $X$. Here $c_1, \cdots, c_{p-1}$ are non-zero scalars. Let $\hat{\beta}$ be the LSE when regressing $Y$ on $X$, and $\tilde{\beta}$ be the LSE when regressing $Y$ on $Z$. Show that $X \hat{\beta} = Z \tilde{\beta}$. (This result implies that the predicated value $\hat{Y}$ is invariant to a full-rank linear transformation on the design matrix.)

4. Consider the model $Y = X \beta + \epsilon$ where

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$$

with $\epsilon \sim (0, \sigma^2 I_4)$. The observed values are $Y_1 = 1, Y_2 = 3, Y_3 = -2, Y_4 = 2$.

(a) Use each of the three methods we introduced in class to find LSEs of $\beta$. Compare the three LSEs and examine whether or not they are identical to each other? In R, you can use the function "solve" to obtain the inverse of a nonsingular matrix, and the function "ginv" (in the library(MASS)) to obtain a generalized inverse of a singular matrix. Be sure to attach your R code at the end of your homework.
(b) Prove that $\beta_2$ and $\beta_1 + \beta_3$ are estimable but $\beta_1$ is not estimable.

(c) For the three parameters in (c), namely $\beta_2$, $\beta_1 + \beta_3$, and $\beta_1$, can you obtain the BLUE for each of them? For those you can obtain BLUE, calculate their BLUEs and compute their variances.