1. Consider two samples \((Y_{11}, \cdots, Y_{1n})\) and \((Y_{21}, \cdots, Y_{2n})\). For each \(i\), we assume that \(Y_{1i}\) and \(Y_{2i}\) have the same predictor \(x_i\). Consider the following regression model
\[
Y_{ki} = \beta_k x_i + \epsilon_{ki} \quad (\epsilon_{ki} \sim N(0, \sigma^2), k = 1, 2; i = 1, 2, \cdots, n),
\]
Note that we use two different slope parameters for the two underlying populations. We are interested in testing \(H : \beta_1 = \beta_2\).

(a) Derive the OLSEs of \(\beta_1\) and \(\beta_2\).

(b) Let \(\hat{\beta}_1\) and \(\hat{\beta}_2\) denote the OLSEs of \(\beta_1\) and \(\beta_2\), respectively. Show that the OLSE under \(H\) equals \((\hat{\beta}_1 + \hat{\beta}_2)/2\).

(c) Obtain \(RSS\) and \(RSS_H\) and verify that
\[
RSS_H - RSS = \frac{\sum x_i^2 (\hat{\beta}_1 - \hat{\beta}_2)^2}{2}.
\]

(d) Show that F test statistic equals
\[
F = \frac{(\hat{\beta}_1 - \hat{\beta}_2)^2}{2S^2/\left(\sum x_i^2\right)}.
\]
where \(S^2 = RSS/(2n - 2)\).

2. Consider a balanced two-way ANOVA model in which
\[
Y_{ijk} = \mu + \alpha_i + \beta_j + \delta_{ij} + \epsilon_{ijk}
\]
for \(i = 1, 2, j = 1, 2,\) and \(k = 1, 2,\) where \(\epsilon_{ijk} \sim N(0, \sigma^2)\) and the identifiability constraints
\[
\sum_i \alpha_i = \sum_j \beta_j = \sum_i \delta_{ij} = \sum_j \delta_{ij} = 0
\]
We consider tests of the following null hypotheses
\[
H_A : \alpha_1 = \alpha_2 = 0 \\
H_B : \beta_1 = \beta_2 = 0 \\
H_{AB} : \delta_{ij} = 0 \text{ for all } i = 1, 2; j = 1, 2
\]
Test statistics can be based on sums of squares

\[
SSE = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij.})^2
\]

\[
SSA = \sum_i \sum_j \sum_k (\bar{Y}_{i..} - \bar{Y}_{...})^2
\]

\[
SSB = \sum_i \sum_j \sum_k (\bar{Y}_{.j} - \bar{Y}_{...})^2
\]

\[
SSAB = \sum_i \sum_j \sum_k (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j} + \bar{Y}_{...})^2
\]

Define mean squares

\[
MSA = SSA/(2 - 1), MSB = SSB/(2 - 1), MSAB = SSAB/[(2 - 1) \times (2 - 1)], MSE = SSE/[2 \times 2 \times (2 - 1)]
\]

(a) Show that SSA, SSB, SSAB, and SSE are independent.

(b) Derive the distribution for \(SSA/\sigma^2\) under \(H_A\). Show that under \(H_A\), \(MSA/MSE\) has an \(F\) distribution.

(c) Derive the distribution for \(SSAB/\sigma^2\) under \(H_{AB}\). Show that under \(H_{AB}\), \(MSAB/MSE\) has an \(F\) distribution.

3. Consider a balanced two-way ANOVA model with random effects, i.e.,

\[
Y_{ijk} = \mu + a_i + b_j + (ab)_{ij} + \epsilon_{ijk}
\]

with

\[
i = 1, 2, \quad j = 1, 2, \quad k = 1, \ldots, 3
\]

where \(a_i \sim N(0, \sigma^2_A)\), \(b_i \sim N(0, \sigma^2_B)\), \((ab)_{ij} \sim N(0, \sigma^2_{AB})\), and \(\epsilon_{ijk} \sim N(0, \sigma^2)\) are all independent. Sums of squares are defined in the same way as for the two-way ANOVA model with fixed effects.

(a) What is \(cov(Y)\)?

(b) Give the distributions for SSA, SSAB, and SSE.