Optimal Möbius Transformation for Information Visualization and Meshing

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What are Möbius transformations?

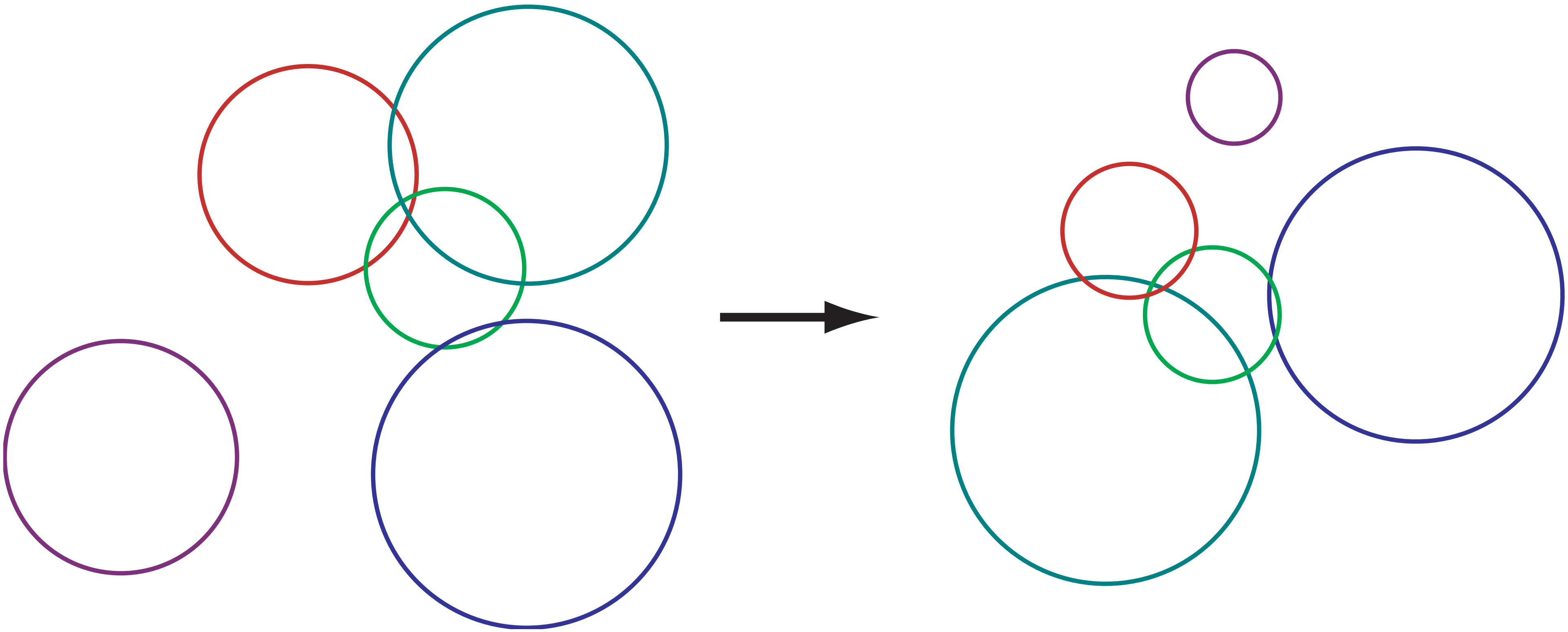
Fractional linear transformations of complex numbers:

$$z \to (a\, z + b) \;/\; (c\, z + d)$$

But what does it mean geometrically?
How to generalize to higher dimensions?
What is it good for?

What are Möbius transformations? (II)

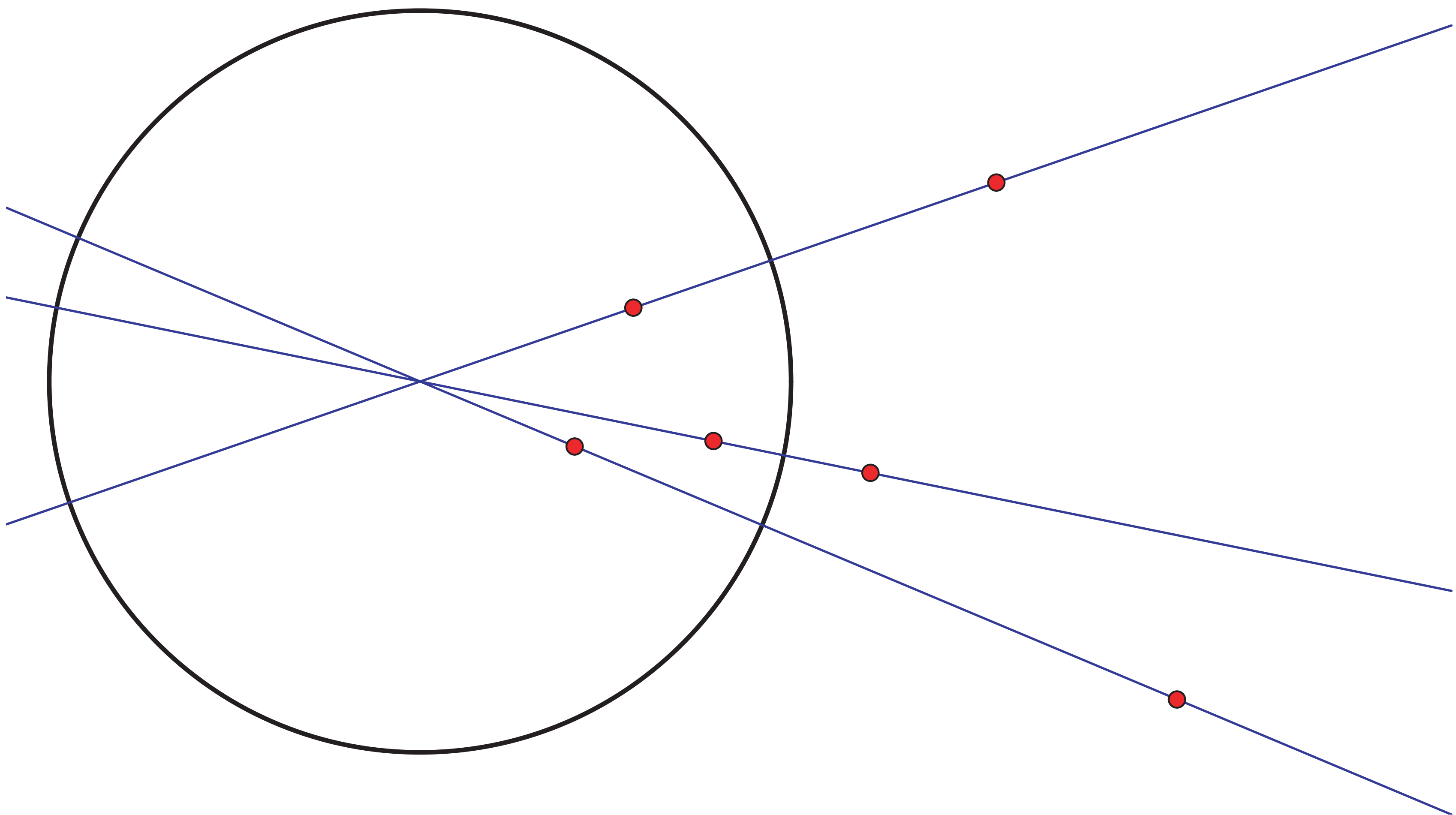
Circle-preserving maps from the plane to itself



More intuitive
Generalizes nicely to spheres, higher dimensional spaces
Not very concrete

What are Möbius transformations (III)

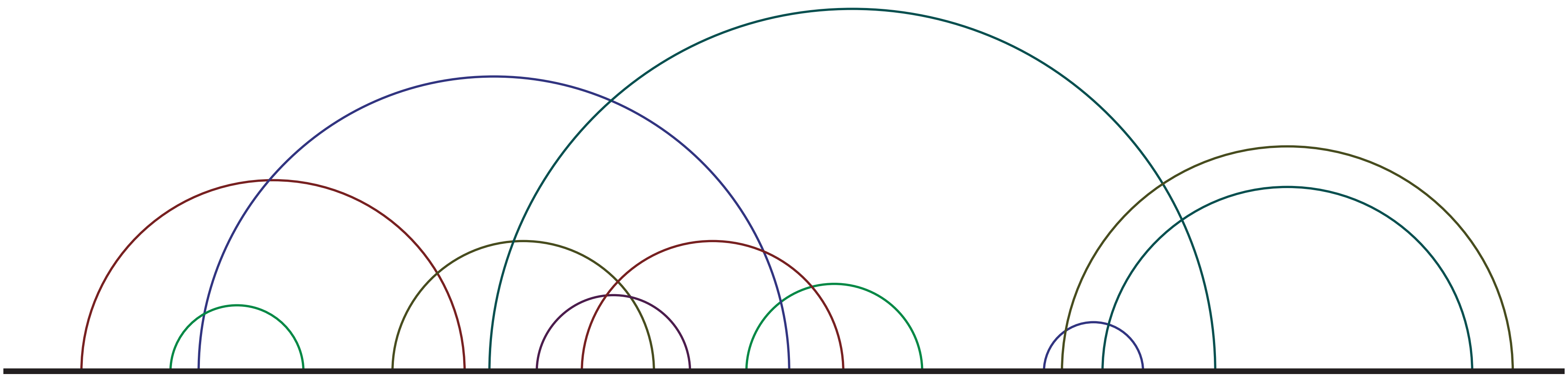
Inversion: map radii of circle to same ray
so that product of distances from center = radius^2



Möbius transformation = composition of multiple inversions
More concrete, still generalizes nicely

What are Möbius transformations? (IV)

View plane as boundary of halfspace model of hyperbolic space



Möbius transformations of plane $\leftrightarrow$ hyperbolic isometries

Esoteric
Most useful for our algorithms

Optimal Möbius transformation:

Given a planar (or higher dimensional) input configuration

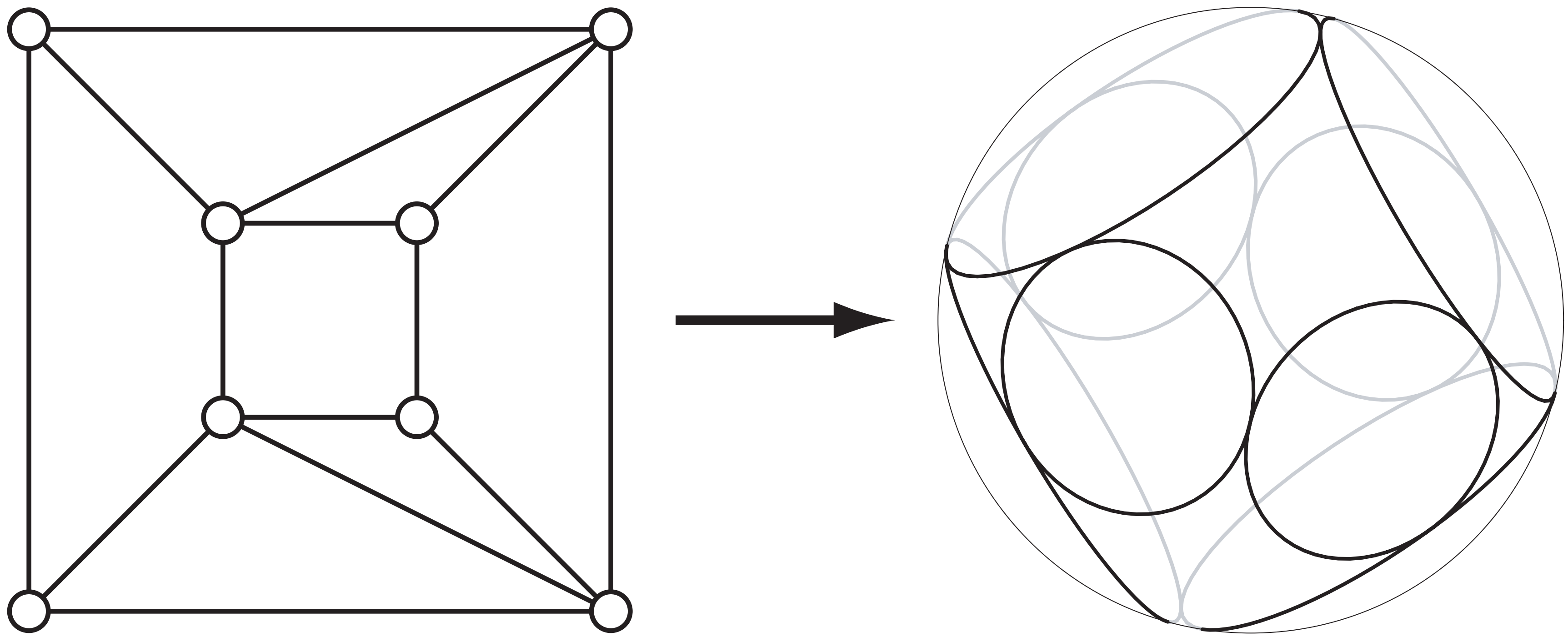
Select a Möbius transformation
from the (six-dimensional or higher) space of all Möbius transformations

That optimizes the shape of the transformed input

Typically min-max or max-min problems:
maximize min(set of functions describing transformed shape quality)

Application: spherical graph drawing

Theorem [Koebe, Thurston]: Any planar graph can be represented
by disjoint disks on a sphere so two vertices adjacent iff two disks tangent



For maximal planar graphs, unique up to Möbius transformation
Other graphs can be made maximal planar by adding vertex in each face

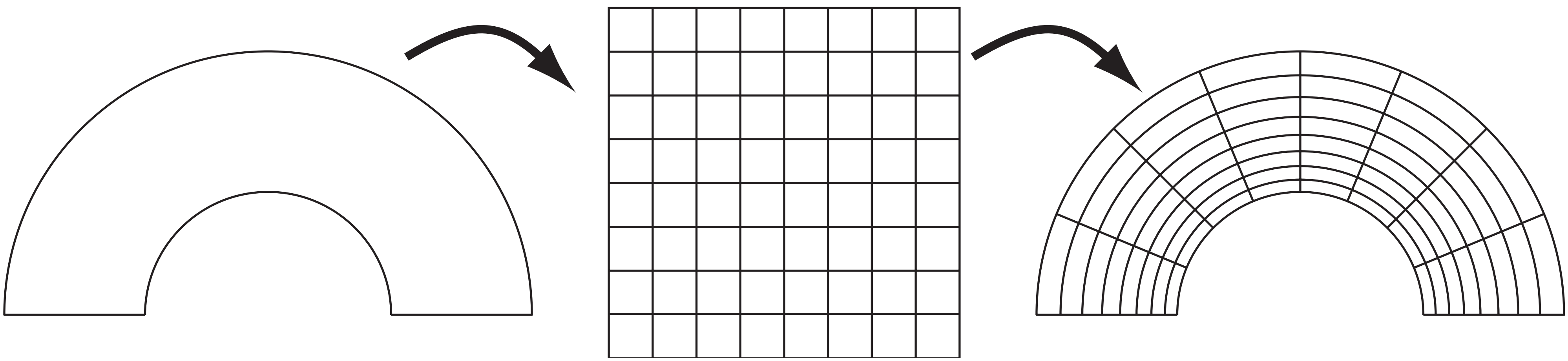
Optimization problem:

Find disk representation of G maximizing minimum disk radius
or, given one disk representation,find Möbius transformation maximizing min radius

Solution also turns out to display all symmetries of initial embedded graph

Application: conformal mesh generation

Given simply-connected planar domain to be meshed
Map to square, use regular mesh, invert map to give mesh in original domain



Different points of domain may have different requirements for element size
To minimize # elements, map regions requiring small size to large areas of square

Conformal map is unique up to Möbius transformation

Optimization Problem:

Find conformal map maximizing min(size requirement * local expansion factor)
to minimize overall number of elements produced

Application: hyperbolic browser [Lamping, Rao, and Pirolli, 1995]

Technique for viewing large web sites or other structured information
by laying out information in hyperbolic space

Allows “fisheye view”: close-up look at details of some point in site
global structure of site visible towards boundary of hyperbolic model

The farther away a point is from the viewpoint (in hyperbolic distance)
the smaller the information it represents will be displayed

Optimization problem:

Find good initial viewpoint for hyperbolic browser
in order to make overall site as visible as possible

Maximize minimum size of displayed object
or
Maximize minimum separation between two objects

Application: brain flat mapping [Hurdal et al. 1999]

Problem: visualize the human brain

All the interesting stuff is on the surface
But difficult because the surface has complicated folds

Approach: find quasi-conformal mapping brain → plane
Then can visualize brain functional units as regions of mapped plane

Avoids distorting angles but areas can be greatly distorted

As in mesh gen. problem, mapping unique up to Möbius transformation

Optimization problem:

Given map 3d triangulated surface → plane,
find Möbius transformation minimizing max(area distortion of triangle)

Optimal Möbius Algorithm

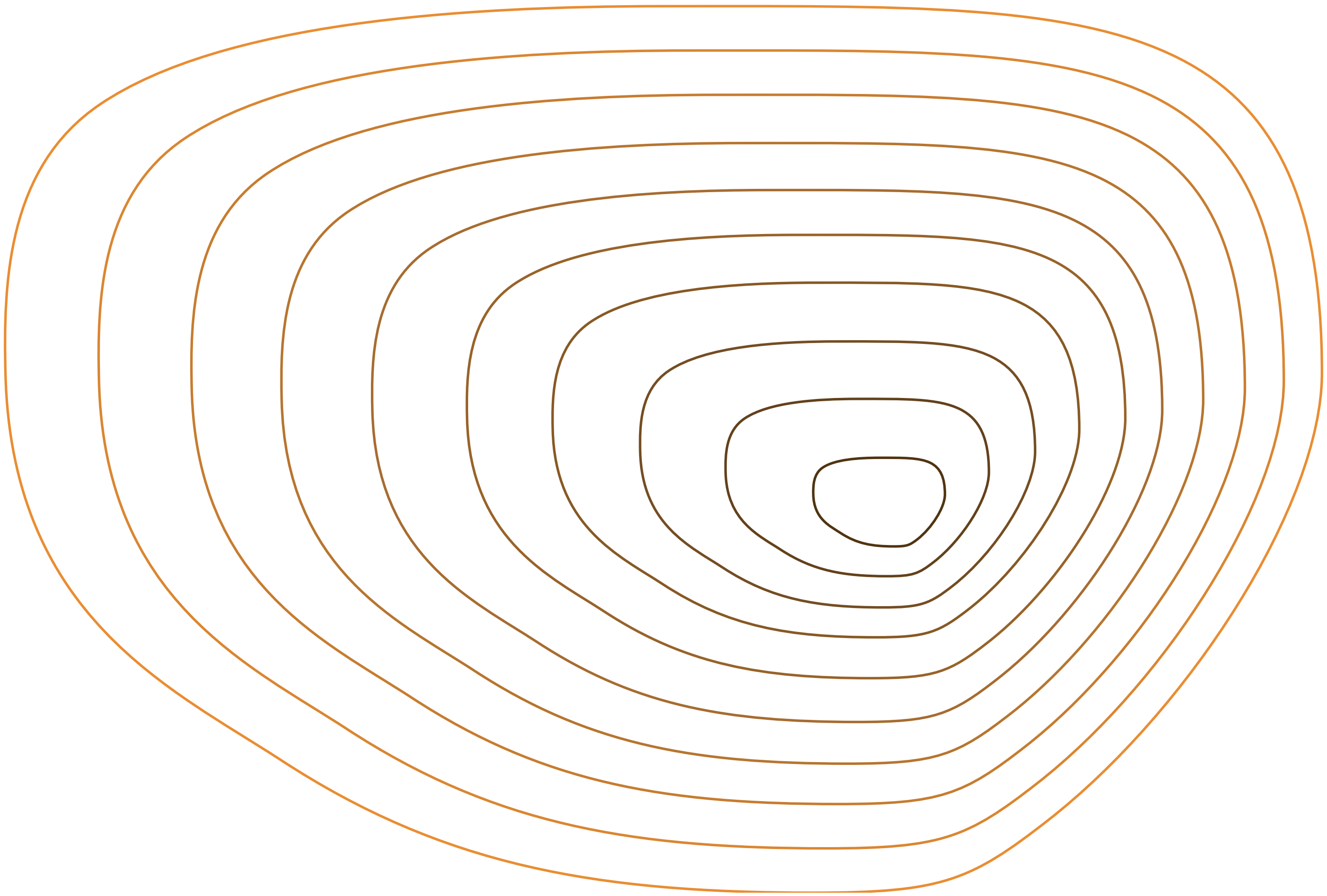
Key components:

Quasiconvex Programming

Hyperbolic Geometry

Quasiconvex functions

Level sets are nested convex curves



Inner curves correspond to smaller function values

(Like topographic map of open pit mine)

Quasiconvex programming [Amenta, Bern, Eppstein 1999]

Given n quasiconvex functions f_i

$\max(f_i(x))$ is also quasiconvex

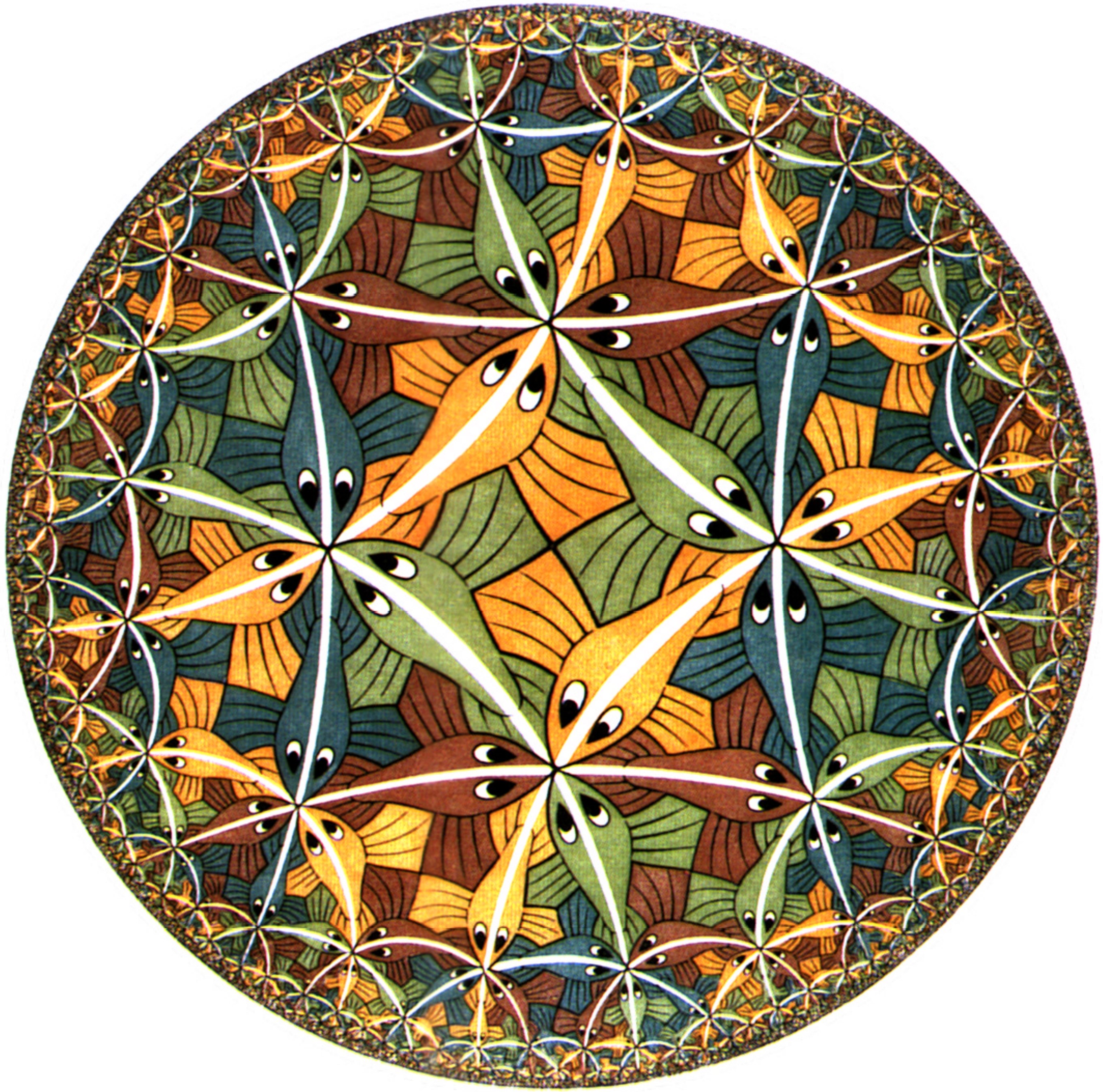
problem is simply to compute x minimizing $\max(f_i(x))$

Can be solved exactly with $O(n)$ constant-size subproblems
using low-dimensional linear-programming-type techniques

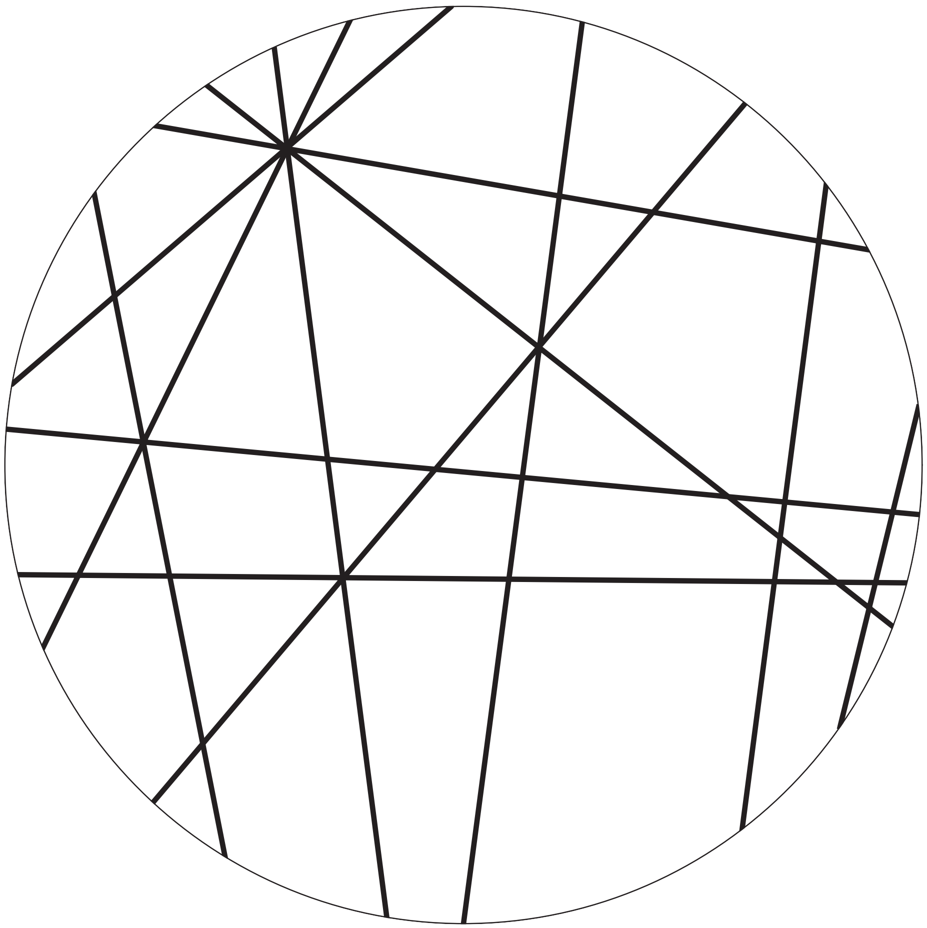
Can be solved numerically by hill-climbing or other local optimization methods

Hyperbolic space (Poincaré model)

Interior of unit sphere; lines and planes are spherical patches perpendicular to unit sphere



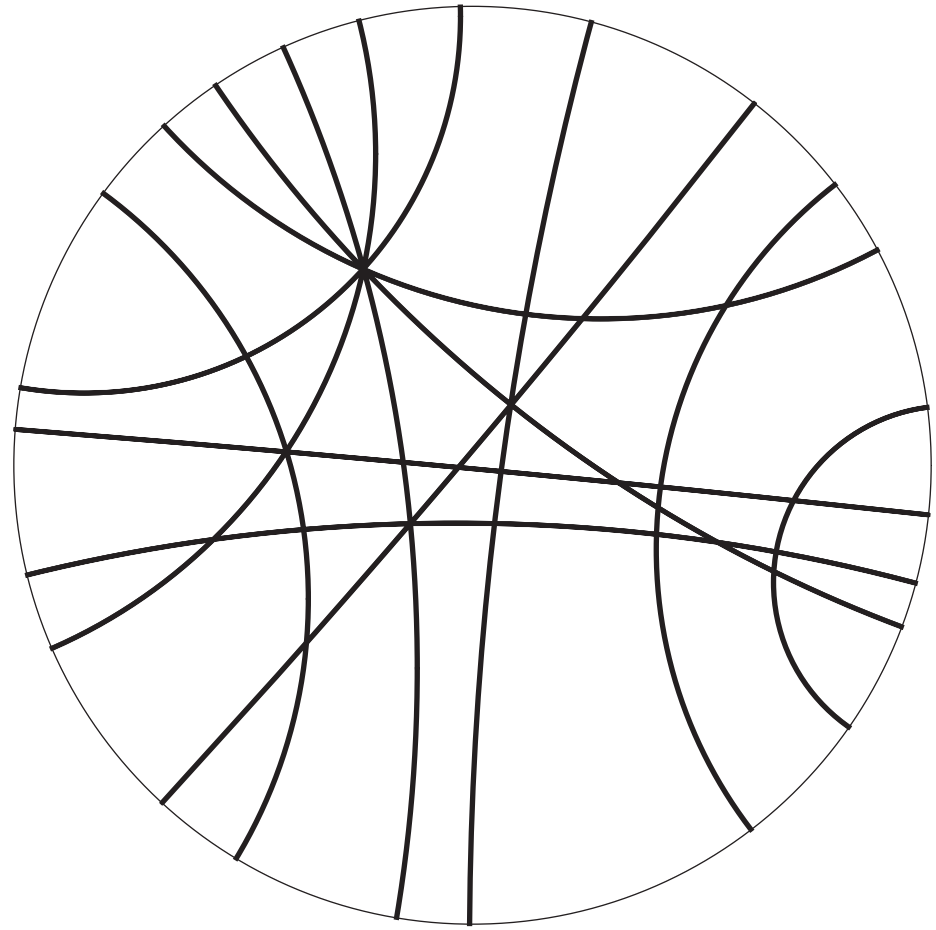
Two models of hyperbolic space



Klein Model

Hyperbolic objects are straight or convex
iff their model is straight or convex
Angles are severely distorted

Hyperbolic symmetries are modeled as
Euclidean projective transformations



Poincaré Model

Angles in hyperbolic space
equal Euclidean angles of their models
Straightness/convexity distorted

Hyperbolic symmetries are modeled as
Möbius transformations

Möbius transformation and hyperbolic geometry

Quasiconvex programming works equally well in hyperbolic space
Due to convexity-preserving properties of Klein model

But space of Möbius transformations is not hyperbolic space...

View Möbius transformation as choice of Poincaré model

Factor transformations into
choice of center point in hyperbolic model (affects shape)
Euclidean rotation around center point (doesn't affect shape)

Optimal Möbius transformation algorithm

Represent optimization problem objective function
as max of set of quasiconvex functions
where function argument is hyperbolic center point location

Hard part: proving that our objective functions are quasiconvex

Solve quasiconvex program

Use center point location to find Möbius transformation

Conclusions

Formulate several interesting applications as Möbius optimization

Can solve via LP-type techniques or hill-climbing

Interesting use of hyperbolic methods in computational geometry

but...

Details of exact algorithm may be difficult to implement
(see Gärtner for similar difficulties in LP-type min-volume ellipsoid)

Not able to prove quasiconvexity in some cases
e.g. given number x, triangle T at infinity in hyperbolic 3-space
are points from which T subtends solid angle $> x$ convex?