Genus, Treewidth, and Local Crossing Number

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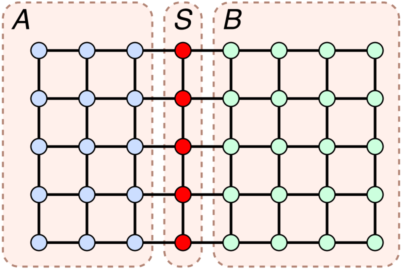
Planar graphs have many nice properties

- They have nice drawings (no crossings, etc.)
- They are sparse (# edges $\leq 3n - 6$)
- They have small separators, or equivalently low treewidth (both $O(\sqrt{n})$, important for many algorithms)



But many real-world graphs are non-planar

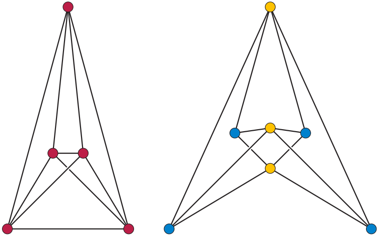
Even road networks, defined on 2d surfaces, typically have many crossings [Eppstein and Goodrich 2008]

Almost-planarity

Find broader classes of graphs
defined by having nice drawings

(bounded genus,
few crossings/edge,
right angle crossings, etc.)

Prove that these graphs still
have nice properties
(sparse, low treewidth, etc.)



RAC drawings of K_5 and $K_{3,4}$

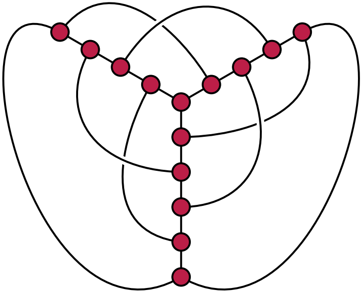
k-planar graph properties

k-planar: $\leq k$ crossings/edge

$\#$ edges $= O(n\sqrt{k})$
[Pach and Tóth 1997]
$\Rightarrow O(nk^{3/2})$ crossings

Planarize and apply planar separator theorem
$\Rightarrow$ treewidth is $O(n^{1/2}k^{3/4})$
[Grigoriev and Bodlaender 2007]

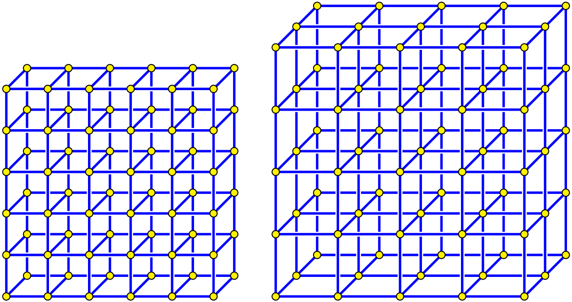
Is this tight?



1-planar drawing of the Heawood graph

Lower bound for k-planar treewidth

$\sqrt{\frac{n}{k}} \times \sqrt{\frac{n}{k}} \times k$ grids are always k-planar



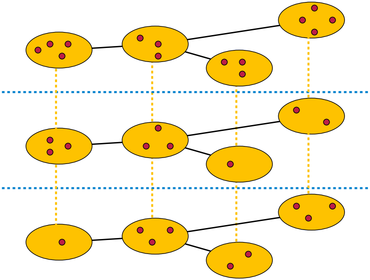
$$\text{Treewidth} = \Omega\left(\sqrt{\frac{n}{k}} \cdot k\right) = \Omega\left(\sqrt{kn}\right) \text{ when } k = O(n^{1/3})$$

Subdivided 3-regular expanders give same bound for $k = O(n)$

Key ingredient: layered treewidth

Partition vertices into layers such that, for each edge, endpoints are at most one layer apart

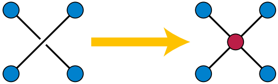
Combine with a tree decomposition
(tree of bags of vertices, each vertex in contiguous subtree of bags, each edge has both endpoints in some bag)



Layered width = maximum intersection of a bag with a layer

Upper bound for k-planar treewidth

- Planarize the given k-planar graph G

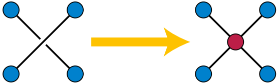


- Planarization's layered treewidth is ≤ 3 [Dujmović et al. 2013]
- Replace each crossing-vertex in the tree-decomposition by two endpoints of the crossing edges
- Collapse groups of $(k+1)$ consecutive layers in the layering
- The result is a layered tree-decomposition of G with layered treewidth $\leq 6(k+1)$
- Treewidth $= O(\sqrt{n \cdot \mathrm{ltw}})$ [Dujmović et al. 2013] $= O(\sqrt{kn})$.

k-Nonplanar upper bound

Suppose we combine k-planar and bounded genus by allowing embeddings on a genus-g surface that have $\leq k$ crossings/edge?

- Replace crossings by vertices (genus-g-ize)



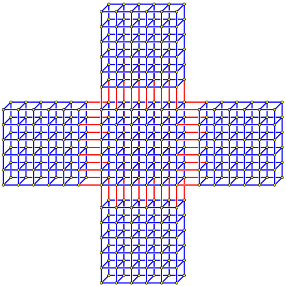
- Genus-g layered treewidth is $\leq 2g + 3$ [Dujmović et al. 2013]
- Replace each crossing-vertex in the tree-decomposition by two endpoints of the crossing edges
- Collapse groups of $(k + 1)$ consecutive layers in the layering
- The result is a layered tree-decomposition of G with layered treewidth $O(gk)$
- Treewidth $= O(\sqrt{n \cdot \text{ltw}}) = O(\sqrt{gkn})$.

k-Nonplanar lower bound

Find a 4-regular expander graph with $O(g)$ vertices

Embed it onto a genus-g surface

Replace each expander vertex by $\sqrt{\frac{n}{gk}} \times \sqrt{\frac{n}{gk}} \times k$ grid



When $n = \Omega(gk^3)$ (so expander edge $\leftrightarrow$ small side of grid) the resulting graph has treewidth $\Omega(\sqrt{gkn})$

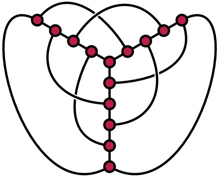
Can sparseness alone imply nice embeddings?

Suppose we have a graph with n vertices and m edges

Then avoiding crossings may require genus $\Omega(m)$
and embedding in the plane may require $\Omega(m)$ crossings/edge

But maybe by combining genus and crossings/edge
we can make both smaller?

$+$



$= ?$

Lower bound on sparse embeddings

For g sufficiently small w.r.t. m,
embedding an m-edge graph on a genus-g surface
may require $\Omega\left(\frac{m^2}{g}\right)$ crossings
[Shahrokhi et al. 1996]

$$\Rightarrow \Omega\left(\frac{m}{g}\right) \text{ crossings per edge}$$

There exist embeddings that get within an $O(\log^2 g)$ factor of this total number of crossings [Shahrokhi et al. 1996]

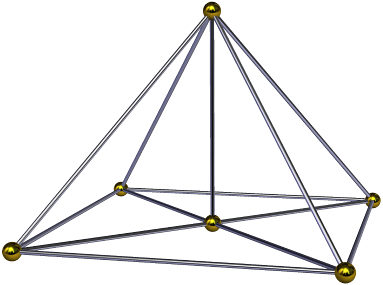
But what about crossings per edge?

Surfaces from graph embeddings (overview)

Embed the given graph G onto another graph H, with:

- Vertex of $G \to$ vertex of H
- Edge of $G \to$ path in H
- Paths are short
- Paths don't cross endpoints of other edges
- Each vertex of H crossed by few paths
- H has small genus edges $-$ vertices $+ 1$

Replace each vertex of H by a sphere and each edge by a cylinder $\Rightarrow$ surface embedding with few crossings/edge



Surfaces from graph embeddings (details)

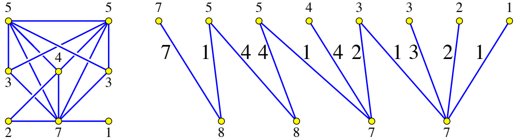
We build the smaller graph H in two parts:

Load balancing gadget

Connects n vertices of G to $O(g)$ vertices in rest of H

Adds $\leq g/2$ to total genus

Groups path endpoints into evenly balanced sets of size $\Theta(m/g)$



Expander graph

Adds $\leq g/2$ to total genus

Allows paths to be routed with length $O(\log g)$ and with $O(m \log g/g)$ paths crossing at each vertex [Leighton and Rao 1999]

Conclusions

n-vertex k-planar graphs have treewidth $\Theta(\sqrt{kn})$

n-vertex graphs embedded on genus-g surfaces with k crossings/edge have treewidth $\Theta(\sqrt{gkn})$

m-edge graphs can always be embedded onto genus-g surfaces with $O\left(\frac{m \log^2 g}{g}\right)$ crossings/edge (nearly tight)

Open: tighter bounds, other properties (e.g. pagenumber), other classes of almost-planar graph, approximation algorithms for finding embeddings with fewer crossings when they exist

References

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