Maximizing the Sum of Radii of Disjoint Balls or Disks

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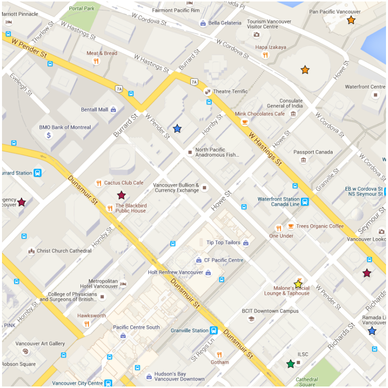
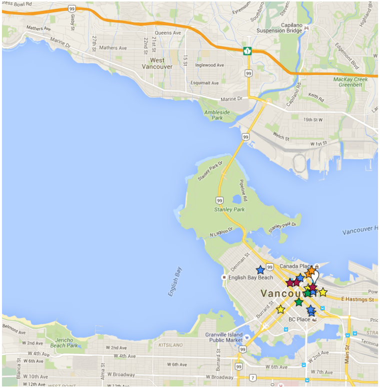
Tradeoff in label size for map labeling





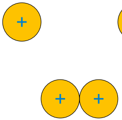
Too small: hard to find among other features

Too big: overlap each other, difficult to separate

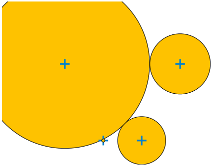
Depends on *local density* more than absolute size

Goal: Find maximum feasible label size

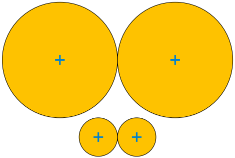
Formally: Place non-overlapping circles with given centers, maximizing some objective function. But what to maximize?



Max min radius: easy (min dist/2) but too global (one close pair makes all circles small)



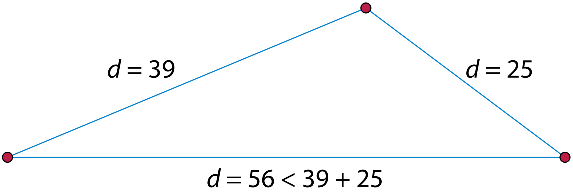
Max total area: too unbalanced, leads to zero-radius circles



Max sum of *radii*: connected circles can stay balanced, disconnected circles vary independently

Detour through abstract metric spaces

Metric space: points with a symmetric non-negative distance function that obeys the triangle inequality: a shortest path from x to y is never longer than a path from x to y passing through z

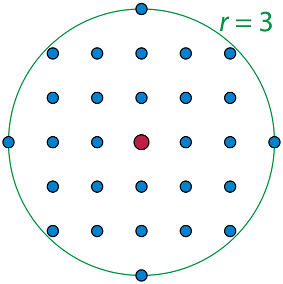


Example: Any finite set of points in $\mathbb{R}^2$ and their distances

Metric balls and when they overlap

Wrong definition: Ball = {points within distance r of center}
Balls overlap when their intersection is nonempty

Difficult to use computationally
Changes when you embed the space into one with more points

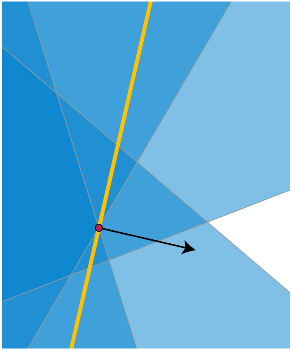


Right definition: Ball = pair (center,radius)
Balls overlap when sum of radii > distance of centers

Metric radius-sum maximization

Given a finite metric space (X, d)
(the circle centers and their distances):

- Choose a radius $r_i \geq 0$ for each center x_i in X
- Obey non-overlapping circle constraints $r_i + r_j \leq d(x_i, x_j)$
- Maximize $\sum r_i$



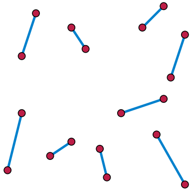
This is a linear program!

...but does it have a combinatorial solution?

Linear programming duality

Every linear program has a *dual* with:

- a variable for each primal constraint
- a constraint for each primal variable
- the same solution value



Our linear program's dual is:

- Find a weight $w_{ij} \geq 0$ for each pair (i, j)
- With each point x_i having total weight $\sum_j w_{ij} \geq 1$
- Minimizing $\sum_{i,j} w_{ij}\, d(x_i, x_j)$

This is the LP relaxation of minimum-length perfect matching on the complete graph of the given center points

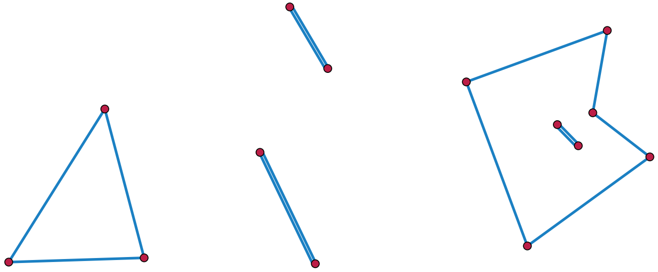
Matching: all weights w_{ij} are 0 or 1; matched edges have weight 1

Relaxation: optimal weights may be 0, 1, or $1/2$

What the dual of our LP actually solves

Choose $2w_{ij}$ edges between each pair of points (x_i, x_j)

The result is the minimum-length 2-regular multigraph over K_n
(a partition of the vertices into odd cycles and 2-cycles)

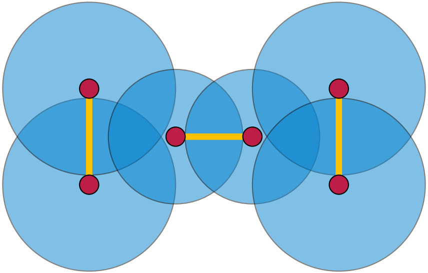


Equivalent (up to unimportant choice of orientation for $>$2-cycles)
to minimum-length matching of the *bipartite double cover*
$K_2 \times K_n$, a graph with two vertices for each input point x_i

From matching back to optimal radii

Most bipartite matching algorithms are *primal-dual*, giving both matched edges and variables of the dual of the LP relaxation

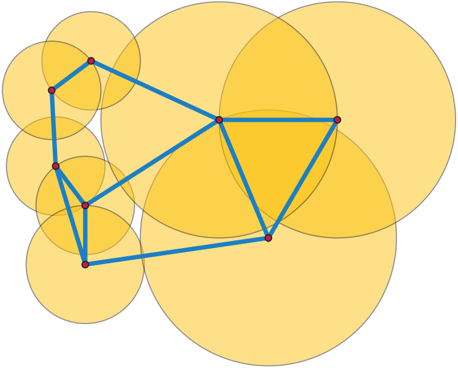
Applying this to matching on $K_2 \times K_n$ gives us two dual variables per vertex: radii of red and blue circles such that each red-blue pair with different centers are non-overlapping



Averaging these two variables gives one optimal radius per center

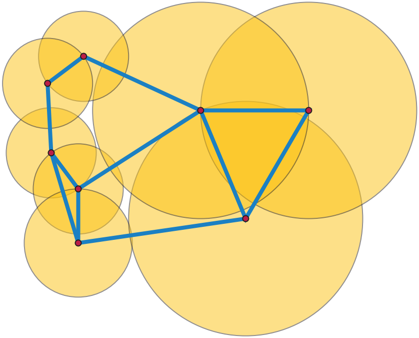
A better graph than the complete graph

We need a supergraph of the optimal 2-regular multigraph
...but it doesn't need to be the complete graph



Instead, use intersection graph of
balls with radius = nearest neighbor distance

Properties of nearest-neighbor intersection graph



- Smallest disk intersects $O(1)$ others
- $\#\text{edges} = O(n)$
- *Separator theorem*: split into constant-factor-smaller pieces by removing $O(n^{1-1/d})$ disks
- Can be constructed in time $O(n \log n)$ (for constant d)

Separator-based weighted bipartite matching

- Construct separator hierarchy
- With separator hierarchy already constructed, shortest paths take linear time [Henzinger et al., JCSS 1997]
- Recursively solve weighted matching for two subgraphs whose intersection is separator and whose union is the whole graph
- For each separator vertex, set dual variable to min from two subproblems and keep matched edge from that subproblem
- Use fast shortest path algorithm to find augmenting paths (≤ 1 per separator vertex) until no more can be found

$$\text{Time} = \text{separator size} \times O(n)$$

Shaves a log from best published bound by Lipton & Tarjan (1980)

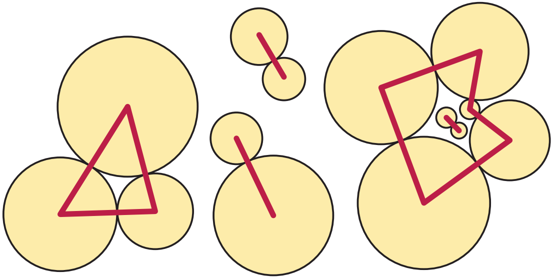
Computes dual variables, not just the matching itself

Putting it all together

Weighted matching on $K_2 \times$ nearest-neighbor intersection graph

Average two dual variables per point to get optimal radii

Time $O(n^3)$ in metric spaces, $O(n^{2-1/d})$ in Euclidean spaces



Optimal solution = odd cycles + pairs of tangent disks