Triangle-Free Penny Graphs: Degeneracy, Choosability, and Edge Count

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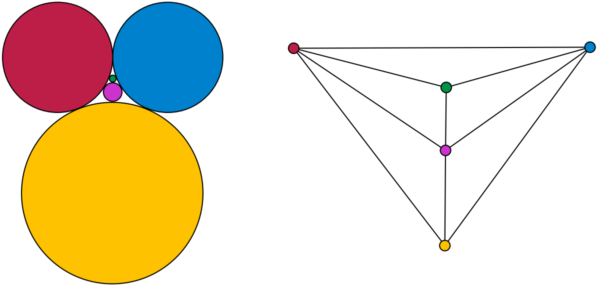
Circle packing theorem

Contacts of interior-disjoint disks in the plane form a planar graph

All planar graphs can be represented this way

Unique (up to Möbius) for triangulated graphs



[Koebe 1936; Andreev 1970; Thurston 2002]

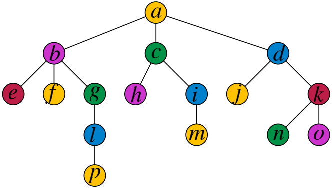
Balanced circle packing

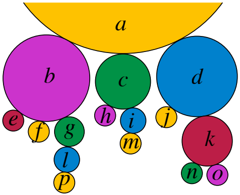
Some planar graphs may require exponentially-different radii

But polynomial radii are possible for:

- Trees
- Outerpaths
- Cactus graphs
- Bounded tree-depth

[Alam et al. 2015]





Perfect balance

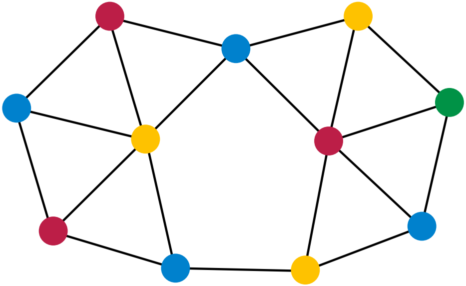
Circle packings with all radii equal represent *penny graphs*

[Harborth 1974; Erdős 1987]

Penny graphs as proximity graphs

Given any finite set of points in the plane

Draw an edge between each closest pair of points



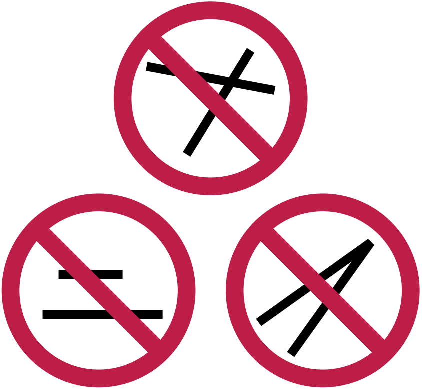
(Pennies: circles centered at the given points
with radius = half the minimum distance)

So penny graphs may also be called *closest-pair graphs*
or *minimum-distance graphs*

Penny graphs as optimal graph drawings

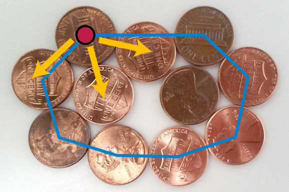
Penny graphs are exactly graphs that can be drawn

- With no crossings
- All edges equal length
- Angular resolution $\geq \pi/3$



Properties of penny graphs

3-degenerate (convex hull vertices have degree ≤ 3)



$\Rightarrow$ easy proof of 4-color theorem; 4-list-colorable
[Hartsfield and Ringel 2003]

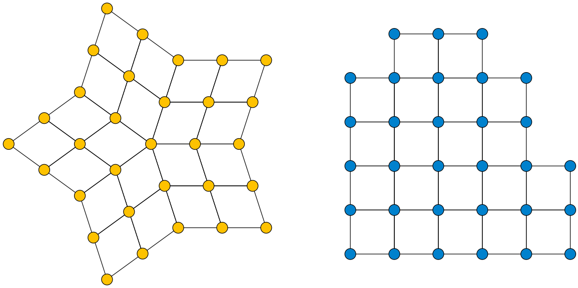
Number of edges at most $3n - \sqrt{12n - 3}$
Maximized by packing into a hexagon
[Harborth 1974; Kupitz 1994]

NP-hard to recognize, even for trees
[Bowen et al. 2015]

Triangle-free penny graphs

Planar equal-edge-length graphs with angular resolution $> \pi/3$

Conjecture [Swanepoel 2009]:
max # edges is $\lfloor 2n - 2\sqrt{n} \rfloor$, given by (partial) square grid



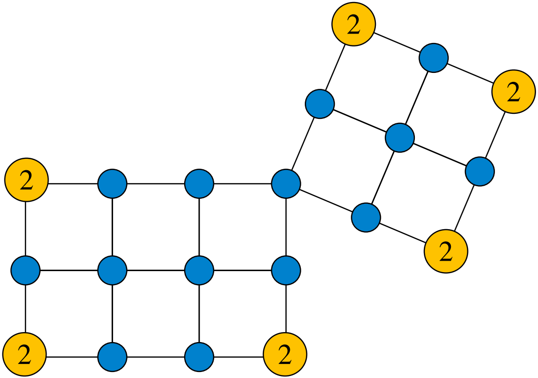
Only known results were inherited from Δ-free planar graphs:

- # edges $\leq 2n - 4$
- 3-colorable [Grötzsch 1959]

New results

2-degenerate (if not tree has ≥ 4 degree-2 vertices)

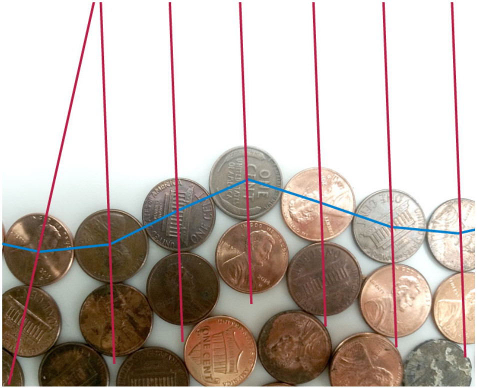
$\Rightarrow$ 3-list-colorable



$\#$ edges $\leq 2n - \Omega(\sqrt{n})$

Proof that some vertices have ≤ 2 neighbors

At each vertex on outer face, draw a ray directly away from neighbor counterclockwise from its clockwise-boundary neighbor



If we walk around boundary, rays rotate by 2π in same direction

But they only rotate positively at vertices of degree ≤ 2!

Proof that # edges $\leq 2n - \Omega(\sqrt{n})$

Isoperimetric theorem:
To enclose area of n pennies, outer face must have $\Omega(\sqrt{n})$ edges

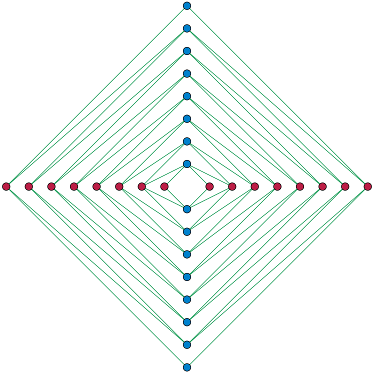
Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

$+$ Algebra with face lengths and Euler's formula

Conclusions and future work

We proved degeneracy and edge bounds for Δ-free penny graphs

The same results hold for *squaregraphs* [Bandelt et al. 2010]
but arbitrary Δ-free planar graphs can be 3-degenerate
or (even when 2-degenerate) have $2n - 4$ edges



Still open: The right constant factor in the $\sqrt{n}$ term

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