Choosing Subsets with Maximum Weighted Average

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October 4, 1996

Abstract

Given a set of n real values, each with a positive weight, we wish to find the subset of $n-k$ values having maximum weighted average. This is equivalent to the following form of parametric selection: given n objects with values decreasing linearly with time, find the time at which the $n-k$ maximum values add to zero. We show that these problems can be solved in time $O(n)$ (independent of k). A generalization in which weights are allowed to be negative is NP-complete.

1 Introduction

A common policy in grading coursework allows students to drop a single homework score. The remaining scores are then combined in some kind of weighted average to determine the student's grade. The problem of performing such calculations automatically has an easy linear time solution: simply try each set of $n-1$ scores. The average for each set can be computed in constant time from the sums of all scores and of all weights.

Consider the generalization of this problem in which not one but two scores may be dropped. More generally, of n scores, suppose k may be dropped. Now how can we maximize the weighted average of the remaining scores, more quickly than the naive $O(n^k)$ algorithm?

We formalize the problem as follows: We are given a set S of n *scores* $\langle v_i, w_i\rangle$, where v_i denotes the *value* of a score and w_i is positive and denotes the *weight* of the score. For later convenience we let V and W denote the sums of the values and weights respectively. We wish to find a subset $T \subset S$, with $|T| = n-k$, maximizing the weighted average

$$A(T) = \frac{\sum_{i\in T} v_i}{\sum_{i\in T} w_i}. \tag{1}$$

In the coursework problem we started with, v_i will typically be the grade received by the given student on a particular assignment, while w_i will denote the number of points possible on that assignment. As we will see, this problem can also be reformulated as a form of *parametric*

selection: given n objects with values decreasing linearly with time, find the time at which the $n - k$ maximum values add to zero.

Some similar problems have been studied before, but we were unable to find any reference to the minimum weighted average subset problem. Algorithms for finding a cycle or cut in a graph minimizing the mean of the edge costs [12, 13, 14, 24] have been used as part of algorithms for network flow [9, 11, 21] and cyclic staffing [16], but the averages used in these problems do not typically involve weights. More recently, Bern et al. [2] have investigated problems of finding the possible weighted averages of a point set in which the weight of each point may vary in a certain range. However that work does not consider averages of subsets of the points.

One intuitively appealing idea for the maximum weighted average problem is a greedy approach: find i_1 maximizing $A(S-i_1)$, i_2 maximizing $A(S-i_1-i_2)$, and so on. Unfortunately this does not always leads to the correct answer. However, there is a confusing assortment of techniques that do correctly solve the problem. From a simple linear-time test for whether a given average is feasible, one can derive algorithms based on binary search, parametric search, or Newton iteration. Our problem can also be expressed as a linear program, to which many algorithms can be applied.

Instead we give several algorithms for the problem that are more efficient than the ones listed above. Our aim is to solve the problem with as good performance as possible (in practice and in theory), but our methods are based on ideas commonly used in computational geometry, so our paper has a secondary point of showing how geometric ideas can be useful even in a problem as seemingly simple as this one.

Our algorithms are based on the following approaches. First, we show that our problem falls into the class of *generalized linear programs*, applied by computational geometers [1, 10, 20] to problems such as the minimum enclosing circle. Generalized linear programming algorithms can be solved in time linear in the input size but exponential or subexponential in the *dimension* of the problem. For our weighted average problem, we show that the dimension is k, so for any fixed k we can find in $O(n)$ time the set of $n - k$ scores with maximum weighted average. A version of Seidel's algorithm [22] provides a particularly simple randomized algorithm for the case $k = 2$; our analysis predicts this to have expected running time approximately three times that of the algorithm for the easy case $k = 1$.

We believe that this generalized linear programming approach will provide the best performance for the grading application we began with, in which k is likely to be only as small as 2 or 3. For larger values of k, this method quickly becomes impractical. Instead, we give another randomized algorithm, based on the linear time quickselect algorithm for finding the k smallest values in a set. The idea is to solve the parametric selection problem defined above by simulating quickselect on the values of the objects as measured at the optimal time t^* we are seeking. We do not have access to these values themselves but we can perform approximate comparisons on them which are sufficient for the selection process. Our method runs in $O(n)$ expected time for any k, and this method appears to improve on our modification of Seidel's algorithm for $k \geq 4$.

This provides a practical solution for moderate to large values of k. Since our algorithm

is randomized, there remains a theoretical question: how quickly can one solve the problem deterministically? We derandomize our algorithm by applying a second technique from computational geometry, *ϵ-cuttings*. The resulting linear-time algorithm settles the theoretical question of the asymptotic complexity of the problem and demonstrates once again the usefulness of geometry in seemingly non-geometric problems.

Finally, we show that if we generalize the problem to allow negative weights, it becomes NP-complete, so no polynomial time solution is likely.

2 Feasibility testing and parametric selection

Suppose that some $(n-k)$-element set $T \subset S$ has weighted average at least A. We can write this as an inequality of the form

$$A \leq A(T) = \frac{\sum_{i \in T} v_i}{\sum_{i \in T} w_i}. \tag{2}$$

Rearranging terms, and using the assumption that the w_i are positive, we can rewrite this as

$$\sum_{i \in T} (v_i - A w_i) \geq 0. \tag{3}$$

Similar inequalities hold if we wish to know whether some T has average strictly greater than A.

Define for each i the function $f_i(t) = v_i - t w_i$, and define

$$F(t) = \max_{|T|=n-k} \sum_{i \in T} f_i(t). \tag{4}$$

Then $F(t)$ can be computed in linear time simply by selecting the $n-k$ largest (or equivalently k smallest) values $f_i(t)$. Equation 3 can be rephrased as saying that some set has average at least A iff $F(A) \geq 0$. We state this as a lemma:

Lemma 1. *For any value A, some set T with $|T| = n - k$ has weighted average at least A iff $F(A) \geq 0$, and some T has weighted average greater than A iff $F(A) > 0$. $F(A)$ can be computed in time $O(n)$.*

$F(A)$ is a piecewise linear decreasing function since it is the maximum of $\binom{n}{n-k}$ decreasing linear functions. Thus our original problem, of finding the maximum weighted average among all $n - k$ point sets, can be rephrased as one of searching for the root of $F(A)$. One can then apply various methods for finding roots of functions, including binary search, Newton iteration, and parametric search, to yield algorithms for our original problem. Some of these solutions can be expressed very simply, but we do not describe them here as they have suboptimal performance.

In subsequent sections we show how to use the feasibility testing function $F(A)$ with randomization and more direct forms of parametric search in order to improve the worst case or expected time to linear.

3 Generalized linear programming

Matoušek et al. [20] define a class of *generalized linear programming problems* that can be solved by a number of algorithms linear in the input size and exponential or subexponential in the *combinatorial dimension* of the problem. Their definition of a generalized linear program follows.

We assume we are given some f taking as its argument subsets of some domain S (in our case, subsets of the given set of scores), and mapping these sets to some totally ordered domain such as the real numbers. This function is required to satisfy the following two properties:

- If $Q \subset R \subset S$, $f(Q) \leq f(R)$.
- If $Q \subset R \subset S$, $f(Q) = f(R)$, and s is any element of S, then $f(Q \cup \{s\}) = f(Q)$ iff $f(R \cup \{s\}) = f(R)$.

A *basis* of such a problem is a set $B \subset S$ such that for any proper subset $X \subset B$, $f(X) < f(B)$. The *dimension* of a problem is the maximum cardinality of a basis. The *solution* to the problem is a basis B such that $f(B) = f(S)$.

In our problem, we let S be the set of scores we are given. We define the objective function $f(X)$ as follows.

$$f(X) = \max_{B \subset X, |B|=k} A(S - B) \tag{5}$$

In other words, we consider a constrained version of our weighted average problem, in which the k scores we drop are required to come from set X. If $|X| < k$, we define $f(X)$ to be a special value $(-\infty, |X|)$ less than any real number. The comparison between two such values $(-\infty, x)$ and $(-\infty, y)$ is defined to give the same result as the comparison between x and y.

Any basis must consist of at most k scores, so the dimension of this problem is k. To verify that this is a generalized linear program, we must prove that it satisfies the requirements above.

Lemma 2. *For the pair (S, f) defined as above from our weighted average problem, any sets $Q \subset R$ satisfy $f(Q) \leq f(R)$.*

Proof: For Q and R both having k or more members, this follows immediately since the choices allowed in the maximization defining $f(R)$ are a superset of the choices allowed for $f(Q)$. A simple calculation shows that it also holds for smaller subsets. □

Lemma 3. *For the pair (S, f) defined as above from our weighted average problem, any sets $Q \subset R$ satisfying $f(Q) = f(R)$, and any score $s = \langle v_i, w_i \rangle$, $f(Q \cup \{s\}) = f(Q)$ iff $f(R \cup \{s\}) = f(R)$.*

Proof: If $|Q| < k$, then the assumption that $f(Q) = f(R)$ forces Q and R to be equal, and the lemma follows trivially. Otherwise, there must be some basis $B \subset Q \subset R$ with $f(B) = f(Q) = f(R)$, and $|B| = k$. Suppose that B is non-optimal in $Q \cup \{s\}$ or $R \cup \{s\}$. By

an argument similar to that in Lemma 1, there is some other k-element set P in $Q \cup \{s\}$ or $R \cup \{s\}$ such that

$$\sum_{i \in S-P} f_i(A(S-B)) > 0. \tag{6}$$

Further, we know that

$$\sum_{i \in S-B} f_i(A(S-B)) = 0 \tag{7}$$

by the definition of f_i. Since the sum for $S-P$ is greater than the sum for $S-B$, there must be scores i, j with $i \in P-B$, $j \in B-P$, and $f_i(A(S-B)) < f_j(A(S-B))$. Then $C = B \cup \{i\} - \{j\}$ also satisfies inequality 6, showing that C is a better set to remove than B. We assumed that B was optimal in Q and R, so it must be the case that $i = s$ and C shows that B is non-optimal in both $Q \cup \{s\}$ and $R \cup \{s\}$. Thus B is optimal in $Q \cup \{s\}$ iff it is optimal in $R \cup \{s\}$. □

Corollary 1. *The pair (S, f) given above defines a generalized linear program, and the solution to the program correctly gives us the set of k scores to drop in S to maximize the weighted average of the remaining points.*

Theorem 1. *For any constant k, we can find the maximum weighted average among all $(n-k)$ element subsets of S, in time $O(n)$.*

For instance, using a modification of Seidel's linear programming algorithm [22], we can find the maximum weighted average using at most $2k!\,n$ expected multiplicative operations. For $k = 2$, the expected number of multiplicative operations of this method is at most $3n + O(\log n)$. Thus we should expect the total time for this algorithm to be roughly three times that for the $k = 1$ algorithm. We will later see that this bound compares favorably to the algorithms we describe below when $k = 2$ or $k = 3$, but for larger values of k the factorial term starts to become too large for this method to be practical.

4 Connection with computational geometry

Before describing more algorithms, we first develop some geometric intuition that will help them make more sense.

Recall that we have defined n linear functions $f_i(A)$, one per score. We are trying to find the maximum weighted average A^*, and from that value it is easy to find the optimal set of scores to drop, simply as the set giving the k minimum values of $f_i(A^*)$.

Consider drawing the graphs of the functions $y = f_i(x)$. This produces an arrangement of n non-vertical lines in the xy plane. The value $f_i(A^*)$ we are interested in is just the y-coordinate of the point where line f_i crosses the vertical line $x = A^*$. We do not care so much about the exact coordinates of this crossing—we are more interested in its order relative to the similar crossing points for other lines f_j, as this relative ordering tells us which of i or j is preferable to drop from our set of scores.

By performing tests on different values of A, using Lemma 1, we can narrow the range in which the vertical line $x = A^*$ can lie to a narrower and narrower vertical slab, having as left and right boundaries some vertical lines $x = A_L$ and $x = A_R$. A_L is the maximum value for which we have computed that $F(A_L) > 0$, and A_R is the minimum value for which we have computed that $F(A_R) < 0$. If we use the algorithm of Lemma 1 on some value $A_L < A < A_R$, we can determine the relative ordering of A^* and A; this results in cutting the slab into two smaller slabs bounded by the line $x = A$ and keeping only one of the two smaller slabs. For instance, the binary search algorithm we started with would simply select each successive partition to be the one in which the two smaller slabs have equal widths.

Any two lines f_i and f_j have a crossing p_{ij} unless they have the same slope, which happens when $w_i = w_j$. If it happens that p_{ij} falls outside the slab $[A_L, A_R]$, we can determine immediately the relative ordering of $f_i(A^*)$ and $f_j(A^*)$, as one of the two lines must be above the other one for the full width of the slab. If the two lines have the same slope, one is above the other for the entire plane and a fortiori for the width of the slab. We can express this symbolically as follows.

Lemma 4. *If $A_L \le A^* \le A_R$, $f_i(A_L) \ge f_j(A_L)$, and $f_i(A_R) \ge f_j(A_R)$, then $f_i(A^*) \ge f_j(A^*)$.*

In this formulation, the lemma can be proven simply by unwinding the definitions and performing some algebraic manipulation.

Define $A(i, j)$ to be the x-coordinate of the crossing point p_{ij}. If f_i and f_j have the same slope, $A(i, j)$ is undefined. If we use Lemma 1 to test the relative ordering of A^* and $A(i, j)$, the resulting slab $[A_L, A_R]$ will not contain $p_{i,j}$ and so by the result above we can determine the relative ordering of $f_i(A^*)$ and $f_j(A^*)$. Symbolically $A(i, j) = (v_i - v_j)/(w_i - w_j)$, and we have the following lemma.

Lemma 5. *If $A(i, j) \le A^*$ and $w_i \ge w_j$, or if $A(i, j) \ge A^*$ and $w_i \le w_j$, then $f_i(A^*) \le f_j(A^*)$.*

Again the lemma can be proven purely algebraically.

The algorithms below can be interpreted as constructing slabs containing fewer and fewer crossing points p_{ij}, until we know enough of the relative orderings of the values $f_i(A^*)$ to select the smallest k such values. This will then in turn give us the optimal set of scores to drop, from which we can compute the desired maximum weighted average as the weighted average of the remaining scores.

For instance, one method of solving our problem would be to use binary search or a selection algorithm among the different values of $A(i, j)$. Once we know the two such values between which A^* lies, all relative orderings among the $f_i(A^*)$ are completely determined and we can apply any linear time selection algorithm that uses only binary comparisons. (Each such comparison can be replaced with an application of Lemma 5.) However there are more values $A(i, j)$ than we wish to spend the time to examine. Instead we use more careful approaches that can eliminate some scores as belonging either to the set of k dropped

scores or the remaining set of $n - k$ scores, without first having to know their relative order compared to all other scores.

We note that similar methods have been applied before, to the geometric problem of selecting from a collection of n points the pair giving the line with the median slope [3, 4, 6, 7, 17, 18, 23]. A geometric duality transformation can be used to transform that problem to the one of selecting the median x-coordinate among the intersection points of n lines, which can then be solved by similar techniques to those above, of finding narrower and narrower vertical slabs until no points are left in the slab. The algorithm is dominated by the time to test whether a given x-coordinate is to the right or left of the goal, which can be done in time $O(n \log n)$. In our weighted average problem, the faster testing procedure of Lemma 1 and the ability to eliminate some scores before all pairwise relations are determined allow us to solve the overall problem in linear time.

5 Randomized linear time

We now describe a randomized algorithm which finds the subset with maximum weighted average in linear time, independent of k. The algorithm is more complicated than the ones we have described so far but should improve on e.g. our modification to Seidel's algorithm for $k > 3$.

The idea behind the algorithm is as follows. If we choose a random member i of our set of scores, and let A^* denote the optimal average we are seeking, the position of $f_i(A^*)$ will be uniformly distributed relative to the positions of the other $f_j(A^*)$. For instance i would have a k/n chance of being in the optimal subset of scores to be removed. If we know that it is in this optimal subset, we could remove from our input those j with $f_j(A^*) < f_i(A^*)$ and update k accordingly. Conversely, if we know that score i has to be included in the set giving the maximum weighted average, we would know the same about all j with $f_j(A^*) > f_i(A^*)$ and we could collapse all such scores to their sum. In expectation we could thus reduce the input size by a constant fraction—the worst case would be when $k = n/2$, for which the expected size of the remaining input would be $3n/4$.

To compute the position of $f_i(A^*)$, and to find the scores to be removed or collapsed as described above, would require knowing the relative ordering of $f_i(A^*)$ with respect to all other values $f_j(A^*)$. For any j we could test this ordering in time $O(n)$ by computing $F(A(i, j))$ as described in Lemma 5. We could compute all such comparisons by binary searching for A^* among the values $A(i, j)$ in time $O(n \log n)$, but this is more time than we wish to take. The solution is to only carry out this binary search for a limited number of steps, giving the position of $f_i(A^*)$ relative to most but not all values $f_j(A^*)$. Then with reasonably high probability we can still determine whether or not score i is to be included in the optimal set, and if this determination is possible we will still expect to eliminate a reasonably large fraction of the input.

We make these ideas more precise in the algorithm depicted in Figure 1. Much of the complication in this algorithm is due to the need to deal with special cases, and to the expansion of previously defined values such as $F(A)$ into the pseudo-code needed to compute

```
Random(S, k):
    A_L ← V/W
    A_R ← +∞
    while |S| > 1 do
        choose i randomly from S
        for ⟨v_j, w_j⟩ ∈ S do
            if w_i = w_j then
                σ(i, j) ← v_j − v_i
                A(i, j) ← −∞
            else
                σ(i, j) ← w_i − w_j
                A(i, j) ← (v_i − v_j)/(w_i − w_j)
        E ← {⟨v_j, w_j⟩ | σ(i, j) = 0}
        X ← {⟨v_j, w_j⟩ | A(i, j) ≤ A_L and σ(i, j) > 0}
            ∪{⟨v_j, w_j⟩ | A(i, j) ≥ A_R and σ(i, j) < 0}
        Y ← {⟨v_j, w_j⟩ | A(i, j) ≤ A_L and σ(i, j) < 0}
            ∪{⟨v_j, w_j⟩ | A(i, j) ≥ A_R and σ(i, j) > 0}
        Z ← S − X − Y − E
        repeat
            A ← median{A(i, j) | ⟨v_j, w_j⟩ ∈ Z}
            for ⟨v_j, w_j⟩ ∈ S do f_j(A) ← v_j − A w_j
            F(A) ← ∑(the largest |S| − k values of f_j(A))
            if F(A) = 0 then return A
            else if F(A) > 0 then A_L = A
            else A_R = A
            recompute X, Y, and Z
            if |X| + |E| ≥ |S| − k then
                remove min(|E|, |X| + |E| + k − |S|)
                    members of E from S
                remove Y from S
                k ← k − (number of removed scores)
            else if |Y| + |E| ≥ k then
                collapse min(|E|, |Y| + |E| − k) members of E
                    and all of X into a single score
        until |Z| ≤ n/32
    return v_1/w_1
```

Figure 1. Randomized algorithm for the maximum weighted average.

them.

Let us briefly explain some of the notation used in our algorithm. $F(A)$ and $f_j(A)$ were defined earlier. Let A^* denote the optimal average we seek; then as noted earlier the optimal subset is found by choosing the scores with the $n-k$ largest values of $f_j(A^*)$. $A(i,j)$ is defined as the "crossover point" for which $f_i(A(i,j)) = f_j(A(i,j))$. We initially let $[A_L, A_R]$ give bounds on the range in which A^* can lie. (It would be equally correct to set $A_L = -\infty$ but the given choice $A_L = V/W$ allows some savings in that some crossover points $A(i,j)$ are eliminated from the range.) Note that $A_L \leq A^*$ since removing scores can only increase the weighted average.

In order to compare f_i and f_j at other values of A, we define $\sigma(i,j)$ as a real value that is positive or negative if for $A > A(i,j)$, $f_i(A) < f_j(A)$ or $f_i(A) > f_j(A)$ respectively. We usually compute $\sigma(i,j)$ by comparing w_i and w_j, but we use a different definition if the weights are equal (corresponding to the geometric situation in which lines $y = f_i(x)$ and $y = f_j(x)$ are parallel).

Then set E consists exactly of those scores that have the same value and weight as the random selection $\langle v_i, w_i \rangle$. Set X consists of those scores for which $f_j(A_L) \geq f_i(A_L)$ and $f_j(A_R) \geq f_i(A_R)$, with one inequality strict. In other words these are the scores for which $f_j(A^*)$ is known by Lemma 4 to be greater than $f_i(A^*)$. Similarly Y consists of those scores for which we know that $f_j(A^*) < f_i(A^*)$. Set Z consists of those scores for which the relation between $f_i(A^*)$ and $f_j(A^*)$ is unknown.

If Z were empty, we would know whether score i itself should be included or excluded from the optimal subset, so we could simplify the problem by also removing all of either X or Y. The purpose of the inner loop of the algorithm is to split the range $[A_L, A_R]$ in a way that shrinks Z by a factor of two, so that this simplification becomes more likely.

In order to compare this algorithm to others including our modification of Seidel's algorithm, we analyze the time in terms of the number of multiplicative operations. It should be clear that the time spent on other operations is proportional to this.

In what follows, we make the simplifying assumption that E contains only score i. This is without loss of generality, as the expected time can only decrease if E has other scores in it, for two reasons. First, Z can initially have size at most $|S - E|$. Second, the fact that scores in E are equivalent to score i lets us treat them either as part of X or as part of Y, whichever possibility allows us to remove more scores from our problem.

Lemma 6. *Let n denote the size of S at the start of an iteration of the outer loop of the algorithm. The expected number of scores removed or collapsed in that iteration is at least $49n/256$.*

Proof: Let p be the position of $f_i(A^*) = v_i - A^* w_i$ in the sorted sequence of such values. Then p is uniformly distributed from 1 to n, so with probability at least $7/8$, p differs by at least $n/32$ from 1, k, and n. Consider the case that $p - k \geq n/32$. Then by the end of the inner loop, we will have at least k scores in Y and can collapse anything placed in X during the loop, removing at least $n - p - n/32$ scores. Similarly if $k - p \geq n/32$ we will have at least $n - k$ scores in X and can remove at least $p - n/32$ scores. The worst case happens when

$k = n/2$, when the expected size of X (in the first case) or Y (in the second case) is $7n/32$. Thus we get a total expectation of $49n/256$. □

Lemma 7. *In a single iteration of the outer loop above, the expected number of multiplicative operations performed is at most* $371n/64$.

Proof: Let n denote the size of S at the start of the outer loop. Note that the only multiplicative operations are n divisions in the computation of $A(i, j)$ in the outer loop, and $|S|$ multiplications in the computation of $F(A)$ in the inner loop. The inner loop is executed at most five times per outer loop, so the worst case number of operations per iteration of the outer loop is $6n$.

To reduce this bound we consider the size of S in each iteration of the inner loop. The analysis of the expected size of S in each iteration is very similar to that in Lemma 6, with the $n/32$ bound on Z replaced by the values n, $n/2$, $n/4$, $n/8$, and $n/16$. For the first three of these, we can prove no bound better than n on the expected value of $|S|$. For the iteration in which $|Z| \leq n/8$, we have probability $1/2$ that p has distance $n/8$ from 1, k, and n, and when p is in this range we can expect to remove $n/8$ values, so the expected size of S in the next iteration is $15n/16$. And for the iteration in which $|Z| \leq n/16$, we have probability $3/4$ that p has distance $n/16$ from 1, k, and n, and when p is in this range we can expect to remove $3n/16$ values in iterations up through this one, so the expected size of S in the last iteration is $55n/64$. Adding these expectations gives the overall bound. □

Theorem 2. *The total expected number of multiplicative operations performed in the algorithm above is at most* $1484n/49 + O(1) \approx 30.3\,n$, *and the total expected time is* $O(n)$.

Proof: The time can be expressed as a random variable which satisfies a probabilistic recurrence

$$T(S) \leq 371|S|/64 + T(R) \tag{8}$$

where R is a random variable with expected size $(1-49/256)|S|$. By the theory of probabilistic recurrences [15], the expected value of $T(S)$ can be found using the deterministic recurrence

$$T(n) = 371n/64 + T((1 - 49/256)n) \tag{9}$$

which solves to the formula given in the theorem. □

Although the constant factor in the analysis of this algorithm is disappointingly large, we believe this algorithm should be reasonable in practice for several reasons. First, the bulk of the time is spent in the computations of $F(A)$, since other operations in the inner loop depend only on the size of Z, which is rapidly shrinking. Thus there is little overhead beyond the multiplicative operations counted above. Second, Lemma 7 assumes that initially $Z = S$, however due to bounds on $[A_L, A_R]$ from previous iterations of the outer loop, Z may actually be much smaller than S. Third, the analysis in Lemma 6 assumed a pathologically bad distribution of the positions of $f_j(A^*)$ for $j \in Z$: it assumed that for p close to k these

positions would always be between p and k, while for p far from k these positions would always be on the far side of p from k. In practice the distribution of positions in Z is likely to be much more balanced, and the number of scores removed will be correspondingly greater. Fourth, for the application to grading, many weights are likely to be equal, which helps us in that there are correspondingly fewer values of $A(i,j)$ in the range $[A_L, A_R]$ and fewer multiplications spent computing $A(i,j)$. Fifth, the worst case for the algorithm occurs for $k = n/2$, but in the case of interest to us k is a constant. For small k the number of operations can be greatly reduced, as follows.

Theorem 3. *Let k be a fixed constant, and consider the variant of the algorithm above that stops when $|Z| < n/16$. Then expected number of multiplicative operations used by this variant is at most $580n/49 + O(k) \approx 11.8n$.*

Proof: We mimic the analysis above. With probability 7/8, $n/16 \le p \le 15n/16$. For such p, the expected number of scores removed is $7n/16$. Therefore the expected number of scores left after an iteration of the outer loop is $(1 - (7/8)(7/16))n = (1 - 49/128)n = 79n/128$. The same sort of formula also tells us how many scores are expected to be left after each iteration of the inner loop. As long as $|Z| \ge n/2$ we can't expect to have removed any scores, so the first two iterations have n expected operations each. In the third iteration, $|Z| \le n/4$, and with probability 1/2 $n/4 \le p \le 3n/4$. For p in that range, we would expect to have removed $n/4$ of the scores. Therefore in the third iteration we expect to have $(1 - (1/2)(1/4))n = (1 - 1/8)n = 7n/8$ operations. Similarly in the fourth iteration we expect to have $(1-(3/4)(3/8))n = 21n/32$ operations. We can therefore express our expected number of operations as a recurrence

$$T(n) = 145n/32 + T(79n/128) \tag{10}$$

with the base case that if $n = O(k)$ the time is $O(k)$. The solution to the recurrence is given by the formula in the theorem. □

This method is already better than our modification of Seidel's algorithm when $k = 4$, and continues to get better for larger k, since the only dependence on k is an additive $O(k)$ term in place of the multiplicative $O(k!)$ factor in our modification of Seidel's algorithm.

6 Deterministic linear time

Much recent work in theoretical computer science has focused on the difference between randomized and deterministic computation. From this work, we know many methods of *derandomization*, that can be used to transform an efficient randomized algorithm (such as our linear time algorithm described above) into a deterministic algorithm (sometimes with some loss of efficiency).

In our case, we have an algorithm that selects a random sample (a score $\langle v_i, w_i \rangle$) from our input, and eliminates some other scores by using the fact that the position of $f_i(A^*)$ in the list of all n such values is likely to be reasonably well separated from 1, k, and n. We

would like to find a similar algorithm that chooses this sample *deterministically*, so that it has similar properties and can be found quickly. Since the randomized algorithm is already quite satisfactory from a practical point of view, this derandomization process mainly has significance as a purely theoretical exercise, so we will not worry about the exact dependence on n (e.g. as measured previously in terms of the number of multiplicative operations); instead we will be satisfied with any algorithm that solves our problem in time $O(n)$.

We return to the geometric viewpoint: graph the linear functions $y = f_i(x)$ to form a line arrangement in the xy-plane, which is cut into slabs (represented as intervals of x-coordinates) by our choices of values to test against A^*. We are trying to reduce the problem to a slab $[A_L, A_R]$ containing few crossings of the lines; this corresponds to having many pairs of scores for which we know the preferable ones to drop.

Many similar problems of derandomization in computational geometry have been solved by the technique of ϵ-*cuttings*, and we use the same approach here. Since the technique is primarily of theoretical interest, we merely sketch it, omitting detailed proofs in favor of saving space.

A *cutting* is just a partition of the plane into triangles. If we are given a set of n lines $y = f_i(x)$ in the xy-plane, an ϵ-cutting for those lines is a cutting for which the interior of each triangle is crossed by a small number of the lines, at most ϵn of them. Matoušek [19] showed that an ϵ-cutting involving $O(1/\epsilon^2)$ triangles can be computed deterministically in time $O(n/\epsilon)$, as long as $1/\epsilon < n^{1-\delta}$ for a certain δ. We will be choosing ϵ to be some fixed constant, so the resulting cutting has $O(1)$ triangles and can be computed in linear time.

The idea of our algorithm is as follows. We first compute an ϵ-cutting for the set of lines $y = f_i(x)$ (that is, a certain triangulation of the xy-plane). By binary search, we can restrict the optimal value A^* we are seeking to lie in a range $[A_L, A_R]$ that does not contain the x-coordinate of any triangle vertex. Therefore if we consider the vertical slab $A_L \le x \le A_R$, the edges of triangles in the cutting either cross the slab or are disjoint from it. If a triangle crosses the slab, at most two of its three sides do so, and the top and bottom boundary lines of the triangle are well defined.

For each edge that crosses the slab, we consider the line $y = ax + b$ formed by extending that edge, and pretend that it is of the form $y = f_i(x)$ for some pair (which must be the pair $\langle b, -a\rangle$). We then use this pair to eliminate scores from S similarly to the way the previous randomized algorithm used the pair $\langle v_i, w_i\rangle$. It turns out not to be important that $\langle b, -a\rangle$ might not be in S. (If it is absent from S we have fewer special cases to deal with. However if the input includes many copies of the same score, it may be necessary for $\langle b, -a\rangle$ to be in S.)

Thus for each pair $\langle b, -a\rangle$ found in this way we compute sets X, Y, and Z as before and eliminate either X or Y if the other of the two is large enough. Unlike the previous randomized algorithm, we do not need an inner loop to reduce the size of Z. Instead we use the definition of our ϵ-cutting to prove that $|Z| \le \epsilon n$. Further, for two edges bounding the top and bottom of the same triangle, the corresponding sizes of X and Y differ from one edge to the other by at most ϵn. Therefore at least one of the edges from the ϵ-cutting, when used as a sample in this way, allows us to eliminate a constant fraction of the input.

Theorem 4. *The algorithm sketched above returns the maximum weighted average among all $(n-k)$ element subsets of S, in time $O(n)$.*

Further details and pseudocode of this deterministic algorithm can be found in the technical report version of this paper [8].

7 Negative weights

The methods we have described do not make any assumption about the values v_i, however the non-negativity of the weights w_i was used in Lemma 1 and therefore in all our algorithms.

It is natural to consider a slight generalization of the problem, in which we allow the weights w_i to be negative. The weighted average V/W of a set S is still well defined (as long as we make some consistent choice of what to do when $W = 0$) so it makes sense to seek the $(n-k)$ element subset maximizing this quantity. We can define $F(A)$ as before, and as before we get a convex piecewise linear function. Unlike the previous situation, there might be either zero or two values of A for which $F(A) = 0$. In the situation in which $F(A)$ has two roots, it turns out that our problem can be solved by finding the larger of these two, by minor modifications of the algorithms we showed before. However it is not so obvious what to do when $F(A)$ has no roots.

Unfortunately, this problem turns out to be NP-complete, as we now show. We use a reduction from the *subset sum* problem, which asks whether, given a collection S of positive integers s_i, and a value t, there is a subset of S with sum exactly t.

Theorem 5. *It is NP-complete to find the $(n-k)$ element subset of a collection of scores $\langle v_i, w_i\rangle$ maximizing the weighted average $\sum v_i / \sum w_i$, if one or more of the weights can be negative.*

Proof: Given an instance (S, t) of the subset sum problem, we transform S to the set of scores $\langle 1, 2s_i\rangle$. We also include n additional "dummy" scores $\langle 1, 0\rangle$ and one final score $\langle 1, 1-2t\rangle$. We then ask for the set of $n+1$ scores maximizing the weighted average. We claim that the maximum possible average is $n+1$ exactly when the subset sum problem is solvable.

Note that for any set containing $\langle 1, 1-2t\rangle$, $\sum w_i$ is odd (so nonzero). For any other set of $n+1$ values there must be at least one score $\langle 1, s_i\rangle$, and the sum is positive (so nonzero). Therefore we need not worry about what to do when $\sum w_i$ is zero. Also note that any sum of weights must be an integer, so all weighted averages of subsets of the input are of the form $(n+1)/x$ for some integer x.

Since all values v_i are equal, this weighted average is maximized by finding a set for which $\sum w_i$ is positive and as small as possible. If some set $A \subset S$ has sum exactly t, we construct a set of scores by including all pairs $\langle 1, s_i\rangle$ for s_i in A, together with $\langle 1, 1-2t\rangle$ and enough dummy scores to make the set have $n+1$ scores total. This set of scores then has $\sum w_i = 1$ and weighted average $n+1$.

Conversely, suppose some set of scores has weighted average $n+1$ and therefore $\sum w_i = 1$. Since this sum is odd, it must include the pair $\langle 1, 1-2t\rangle$. Then the remaining weights must sum to $2t$, and the non-dummy scores from this set can be used to construct a set $A \subset S$ with sum exactly t. □

If weights of zero are not allowed, the dummy scores can be replaced by $\langle 1, \epsilon\rangle$ for any sufficiently small ϵ.

Finally, we note that our original maximum weighted average problem can, like subset sum, be solved in pseudo-polynomial time. Suppose all the v_i and w_i are integers. Let $W_{\min}$ and $W_{\max}$ denote the largest and smallest possible sum of weights in any subset of our scores. We use a dynamic program to determine, for each possible sum of weights w in the range $[W_{\min}, W_{\max}]$, the minimum and maximum sum of values $V_{\min}(w)$ and $V_{\max}(w)$ among all sets having that sum of weights. The maximum weighted average can then be found by comparing $V_{\min}(w)/w$ for negative w with $V_{\max}(w)/w$ for positive w. The algorithm takes time $O(n(W_{\max} - W_{\min}))$.

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