Edges and Switches, Tunnels and Bridges

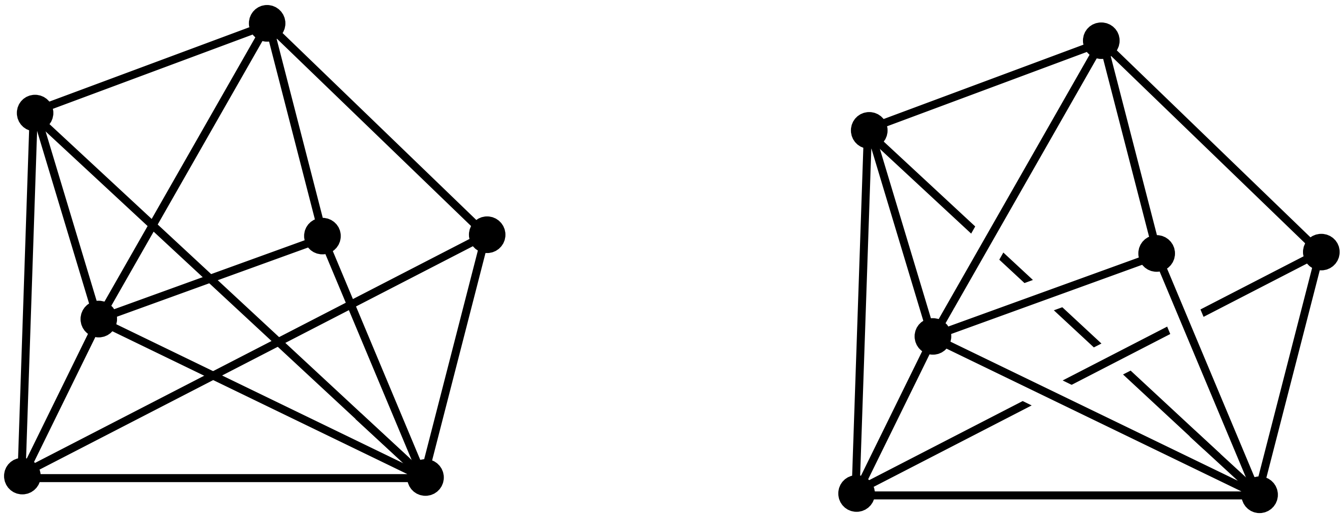
David Eppstein
Marc van Kreveld
Elena Mumford
Bettina Speckmann

Cased drawing

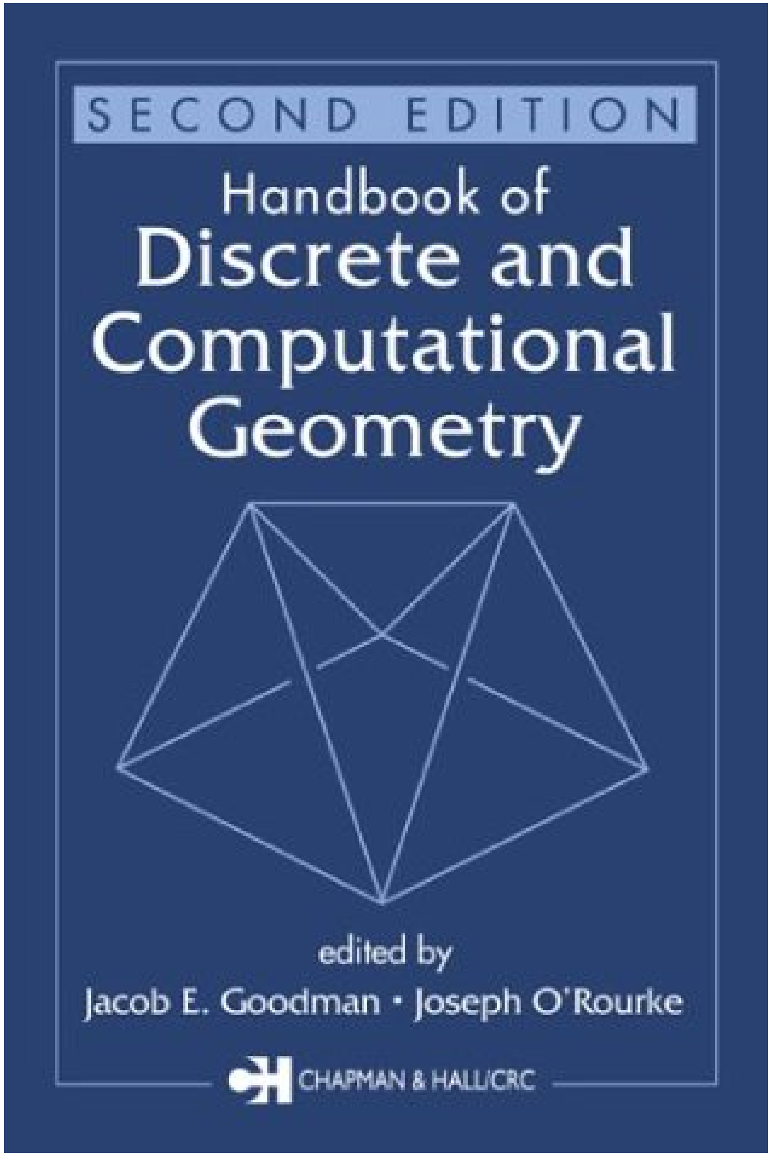
Let D be a non-planar drawing of a graph G.

A cased drawing D' of G is a drawing where

- the edges of each crossing are ordered
- the lower edge is interrupted in an appropriate neighborhood of the crossing



Examples


SECOND EDITION
Handbook of
Discrete and
Computational
Geometry
edited by
Jacob E. Goodman • Joseph O'Rourke
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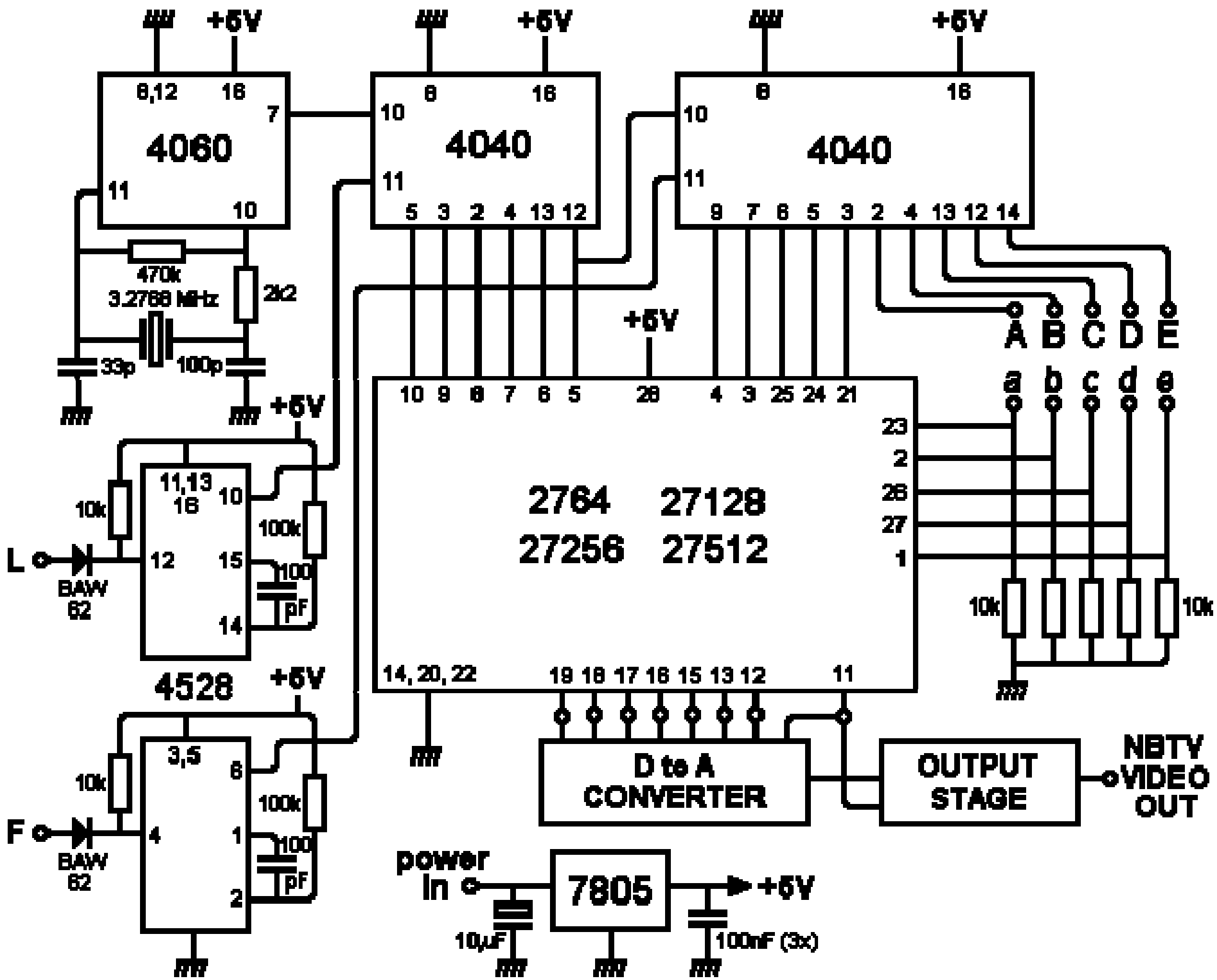
Examples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4040
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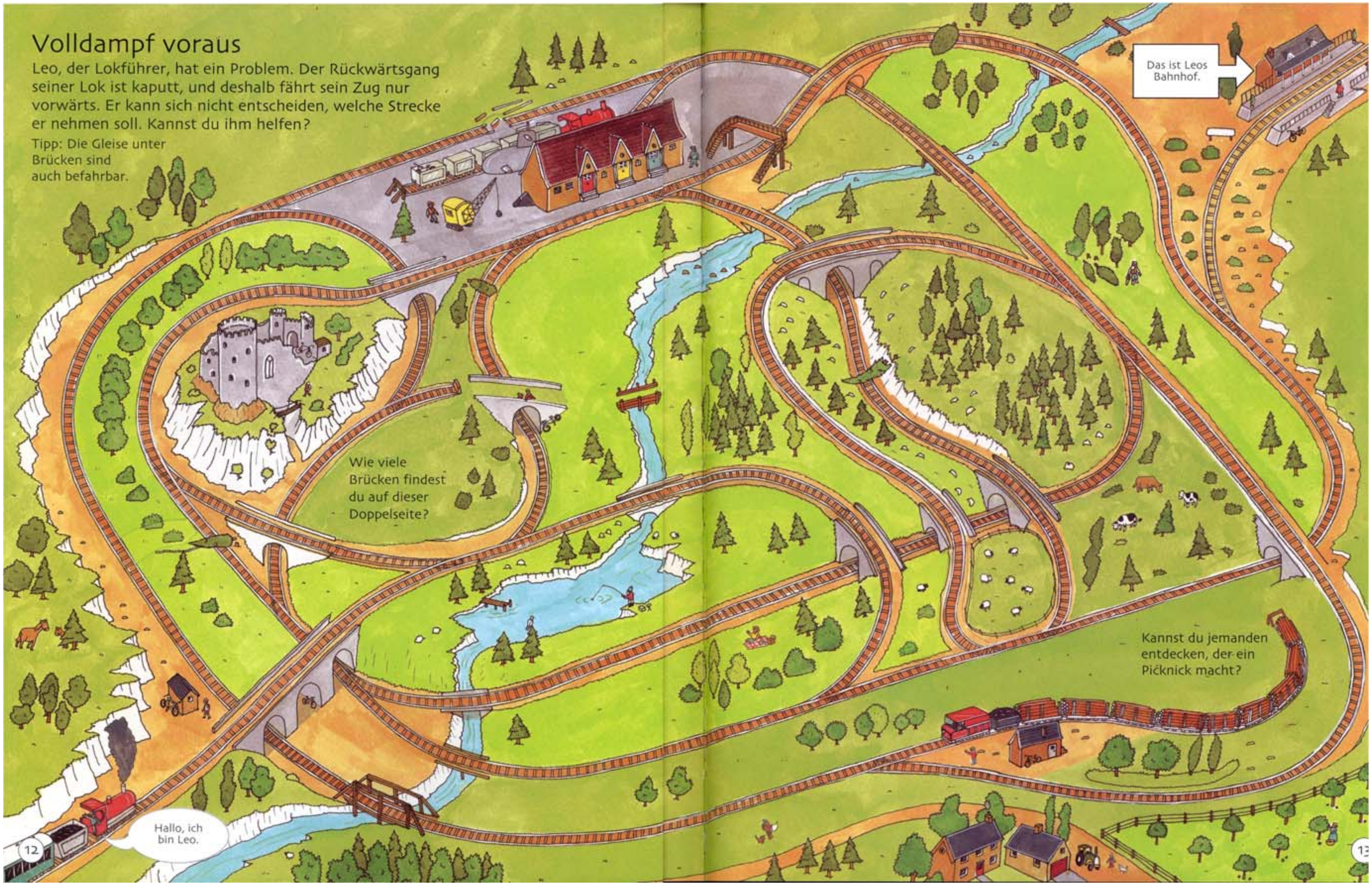
+5V
470k
3.2768 MHz
2k2
33p
100p
10k
100k
100 pF
BAW 62
L
F
4528
2764
27128
27256
27512
A
B
C
D
E
a
b
c
d
e
D to A CONVERTER
OUTPUT STAGE
NBTV VIDEO OUT
power in
7805
10µF
100nF (3x)

Examples



Volldampf voraus
Leo, der Lokführer, hat ein Problem. Der Rückwärtsgang seiner Lok ist kaputt, und deshalb fährt sein Zug nur vorwärts. Er kann sich nicht entscheiden, welche Strecke er nehmen soll. Kannst du ihm helfen?
Tipp: Die Gleise unter Brücken sind auch befahrbar.
Das ist Leos Bahnhof.
Wie viele Brücken findest du auf dieser Doppelseite?
Kannst du jemanden entdecken, der ein Picknick macht?
Hallo, ich bin Leo.
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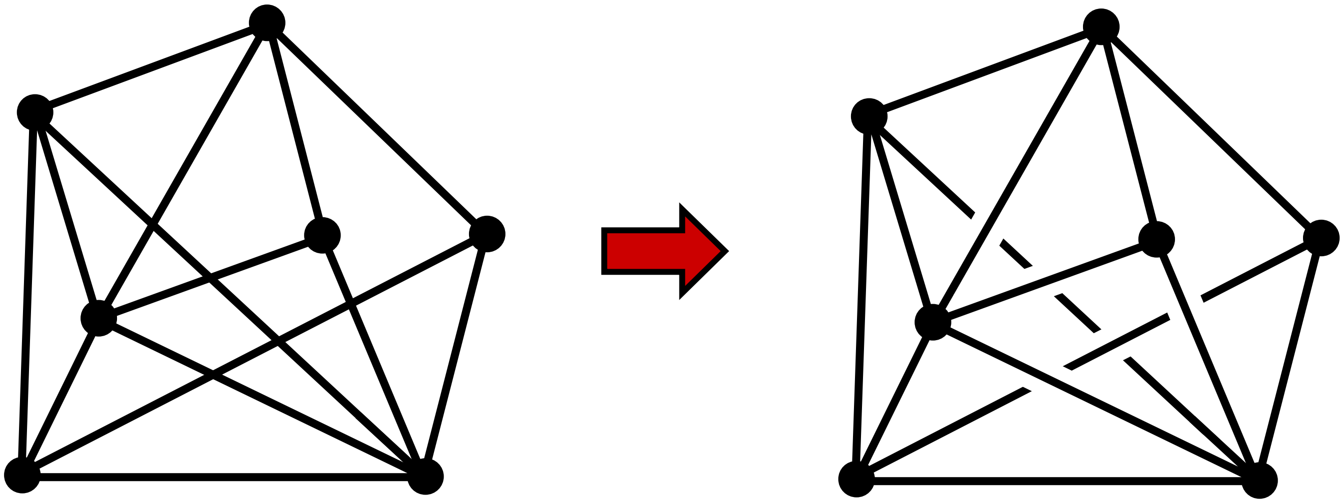
Cased drawing

Given a drawing, turn it into the "best" cased drawing.

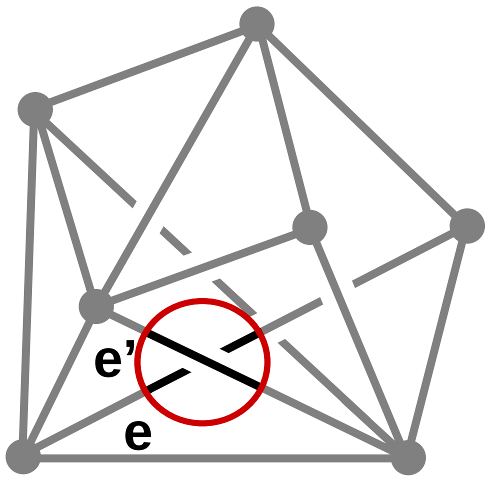


Definitions

A crossing is called

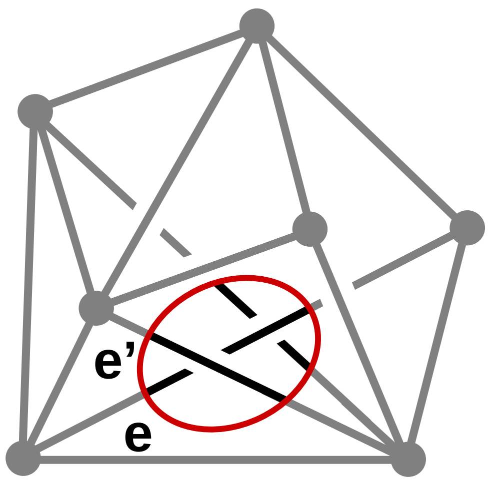
bridge for the edge on top

tunnel for the edge at the bottom



Switch

pair of consecutive crossings along edge e, one a tunnel and the other a bridge for e.



Optimization criteria

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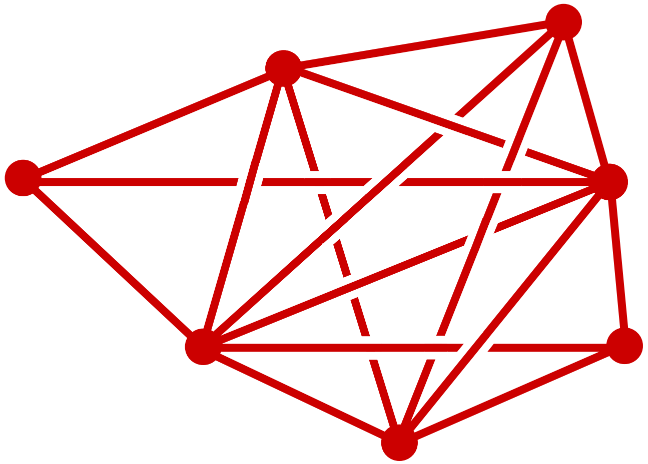
An edge is hard to follow if



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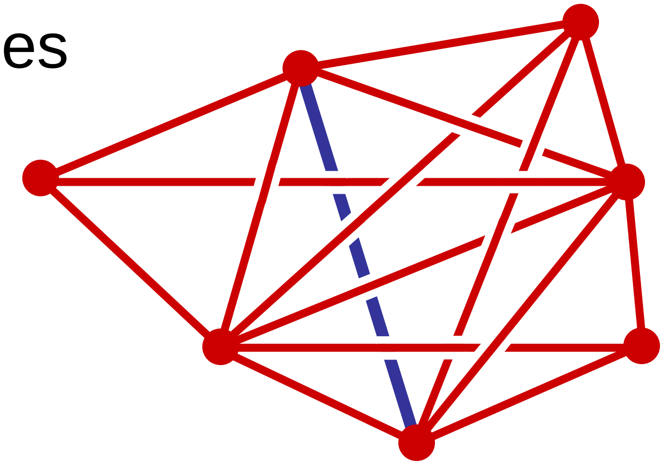
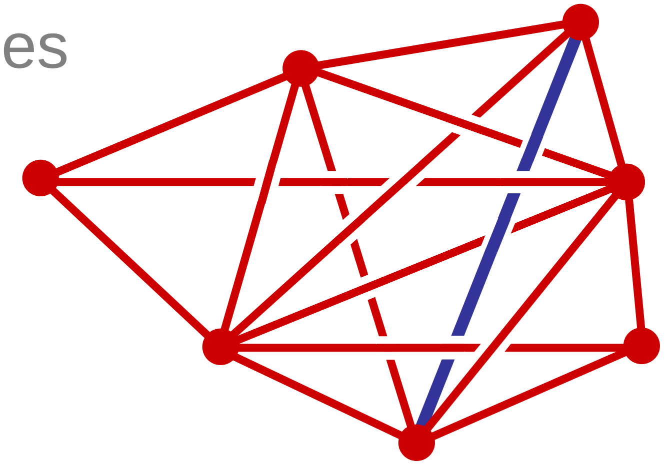


Optimization criteria

An edge is hard to follow if

- it is covered by other edges
- it switches often



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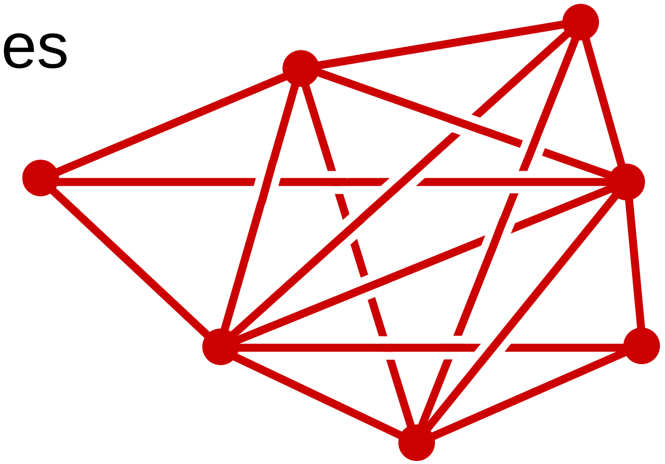

Optimization criteria

An edge is hard to follow if

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  MinMaxTunnels

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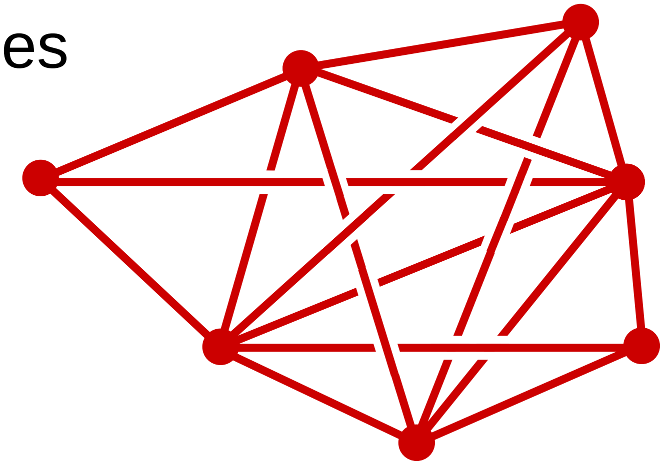

MinMaxTunnels

minimize the number of tunnels per edge.

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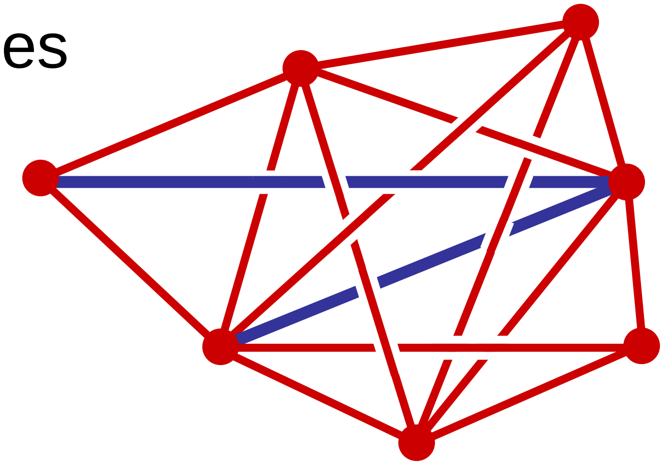

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An edge is hard to follow if

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  MinMaxTunnelLength

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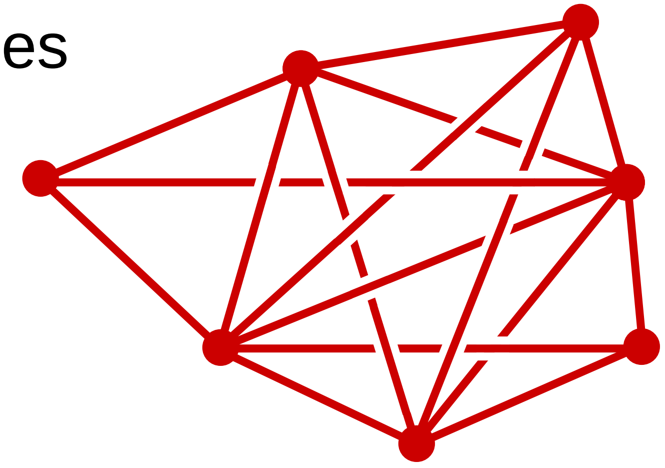

MinMaxTunnelLength

minimize the length of tunnels per edge.

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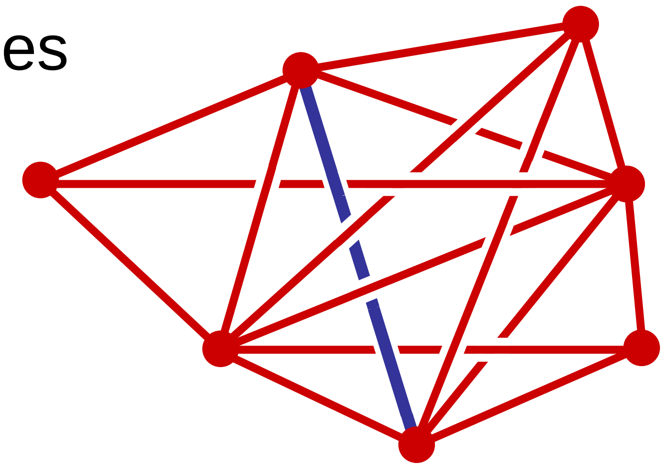

MinMaxTunnelLength

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Optimization criteria

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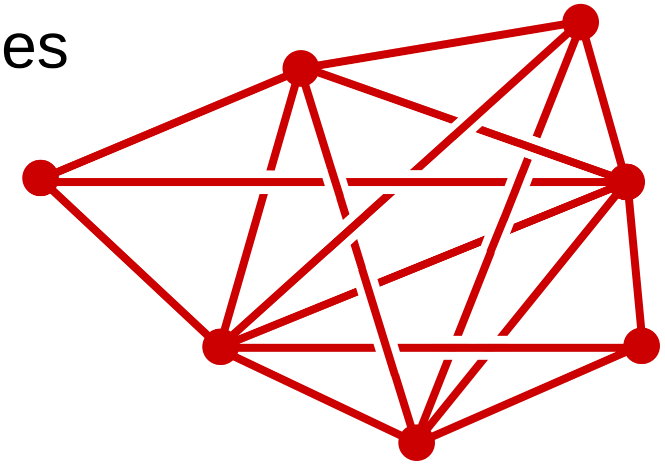
  MinMaxTunnelLength

  MaxMinTunnelDistance



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MaxMinTunnels

maximize the distance between two consecutive tunnels.

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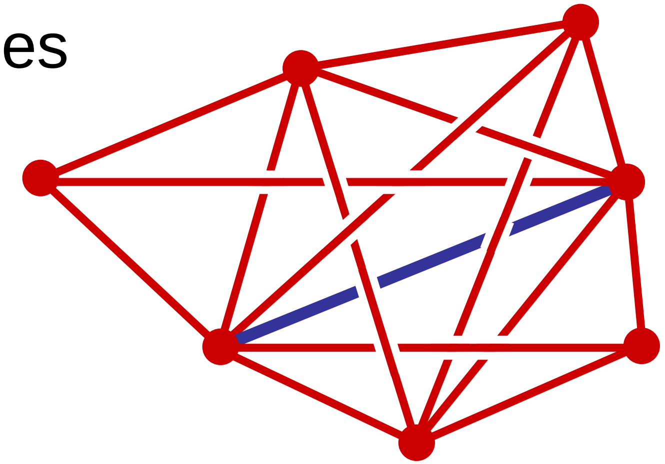

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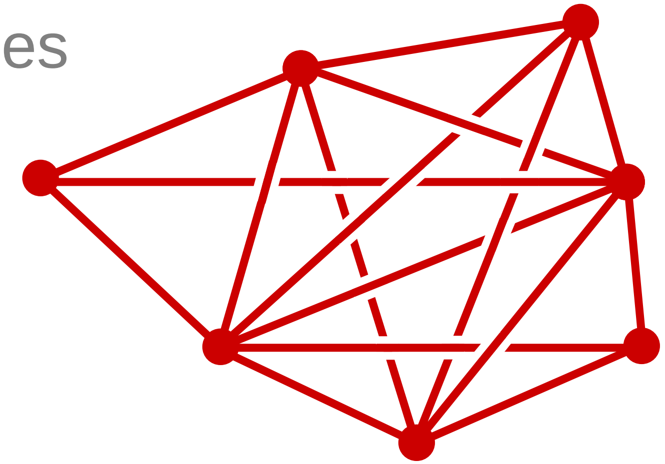

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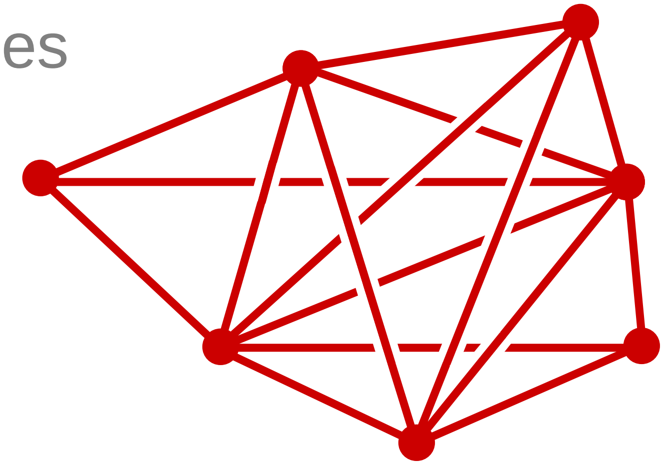

- it switches often

  MinTotalSwitches

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minimize the total number of switches

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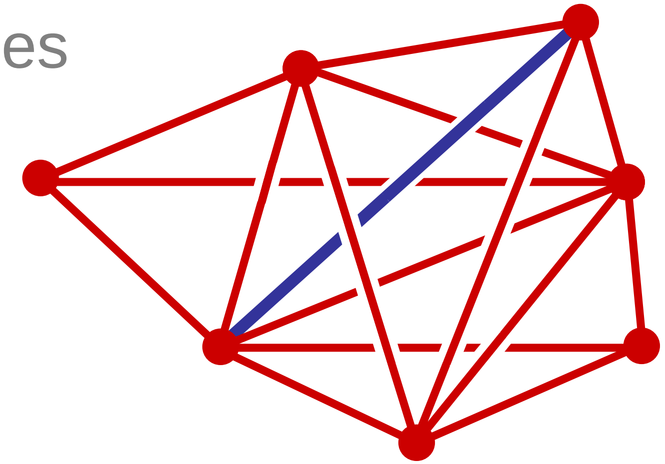

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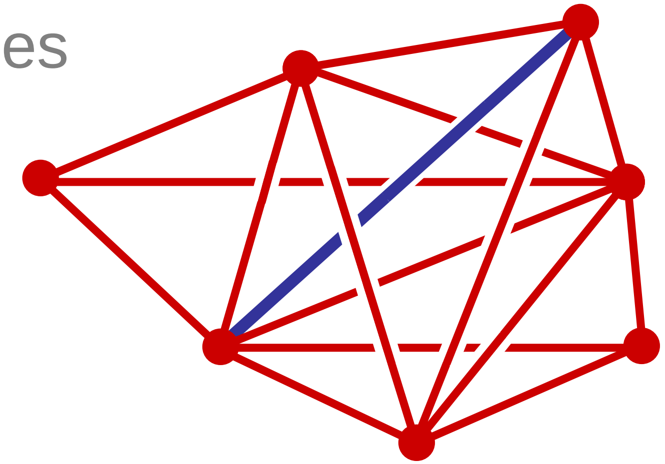


- it switches often

  MinTotalSwitches

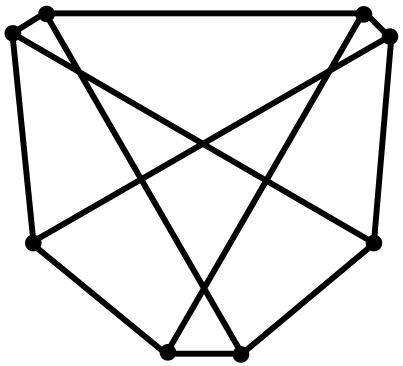
  MinMaxSwitches

MinMaxSwitches

minimize the number of switches per edge

Models

How to define the drawing order?



- weaving
- realizable
- stacking

Models

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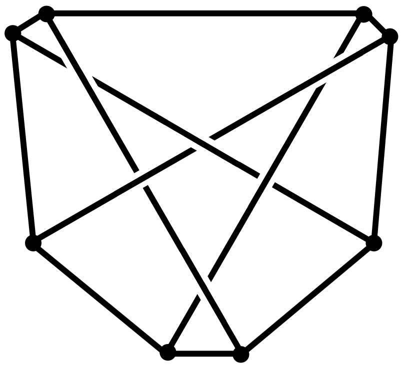
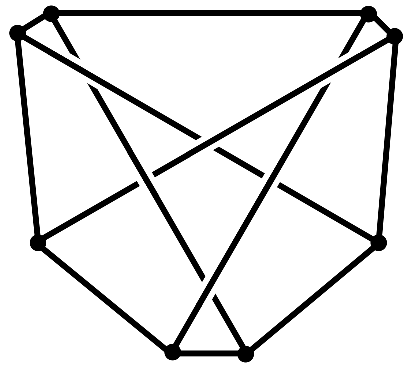


Define drawing order for every crossing separately.

- weaving
- realizable
- stacking

Models: Realizable

How to define the drawing order?

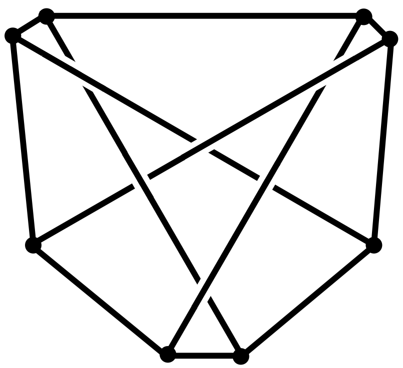


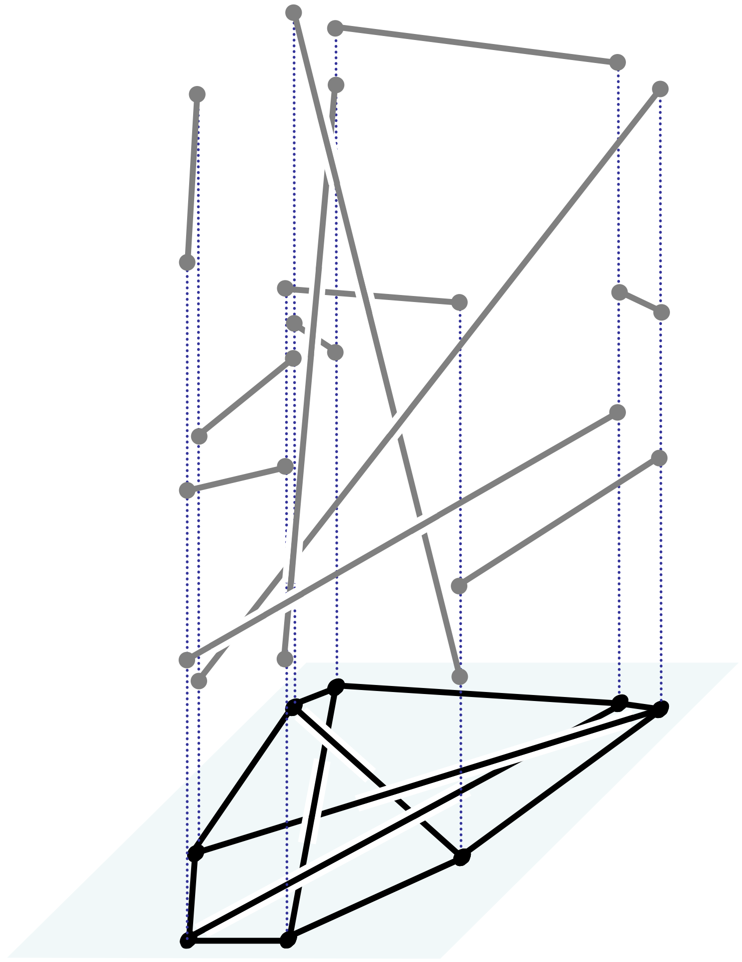
Allow only drawings which are plane projections of line segments in 3 dimensions.

- weaving
- realizable
- stacking

Models: Realizable

How to define the drawing order?





- weaving
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Models: Stacking

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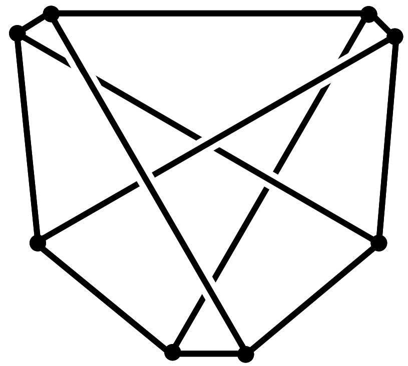

Global top-to-bottom order on edges.

- weaving
- realizable
- stacking

Models: Stacking

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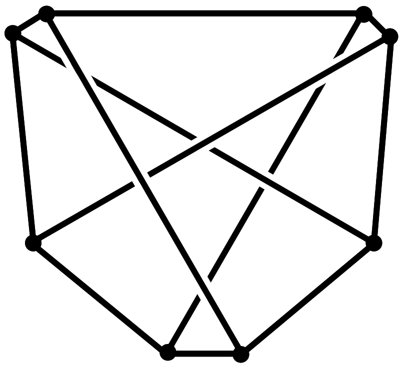

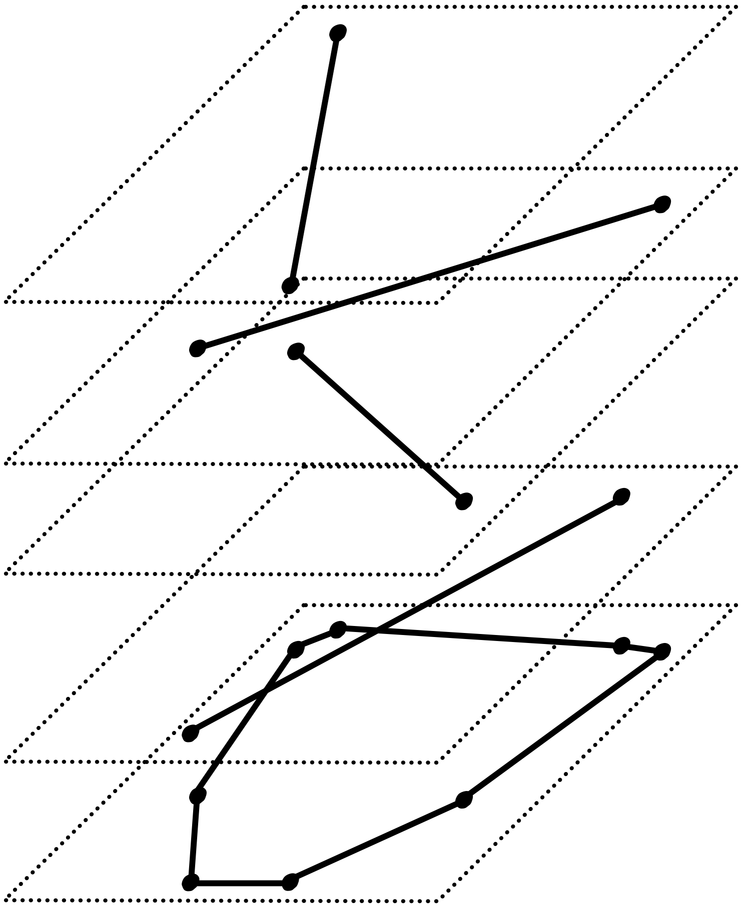

- weaving
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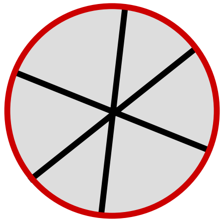
Results

For a drawing D of a graph G with n vertices, m edges,
$k = O(m^2)$ crossings, $q = O(k)$ odd face polygons and
$K = O(m^3)$ total number of pairs of crossings on the same edge

Model	**Stacking**	**Weaving**
MinTotalSwitches	*open*	$O(qk + q^{5/2} \log^{3/2} k)$
MinMaxSwitches	*open*	*open*
MinmaxTunnels	$O(m \log m + k)$ *exp.*	$O(m^4)$
MinMaxTunnelLength	$O(m \log m + k)$ *exp.*	*NP-hard*
MaxMinTunnelDistance	$O((m+k) \log m)$ *exp.*	$O((m+K) \log m)$ *exp.*

Assumptions

- No three edges cross at one point



Theorem

If triple crossings of edges are allowed, then MinTotalSwitches is NP-hard in both the weaving and the stacking model.

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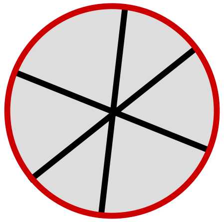
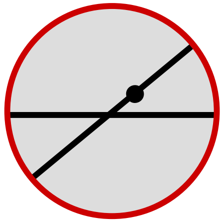


- No vertices on (or very close to) edges



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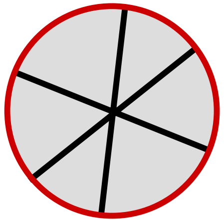

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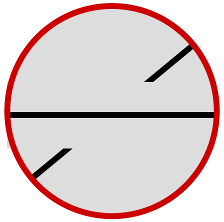

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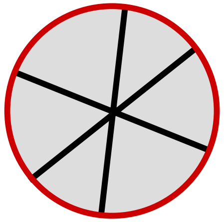

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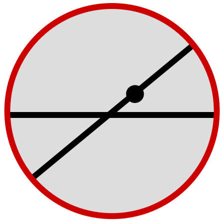
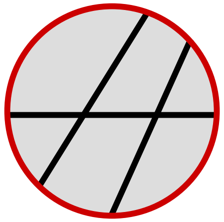


- Edge crossings are well separated



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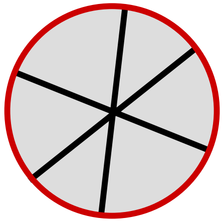

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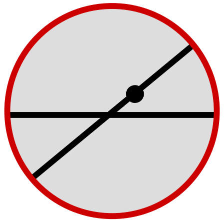

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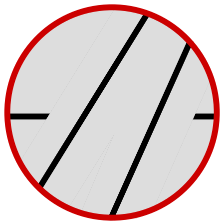

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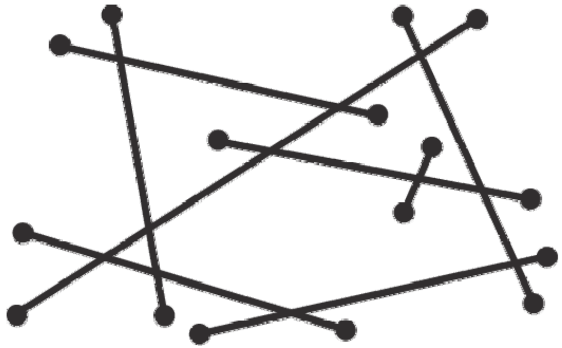
Simplifying the input graph

Lemma

For every graph drawing D of graph G there exist a degree-one graph G' and its drawing D' such that there is one-to one correspondence between

edges of G and G'
casings of D and D'
switches of D and D'



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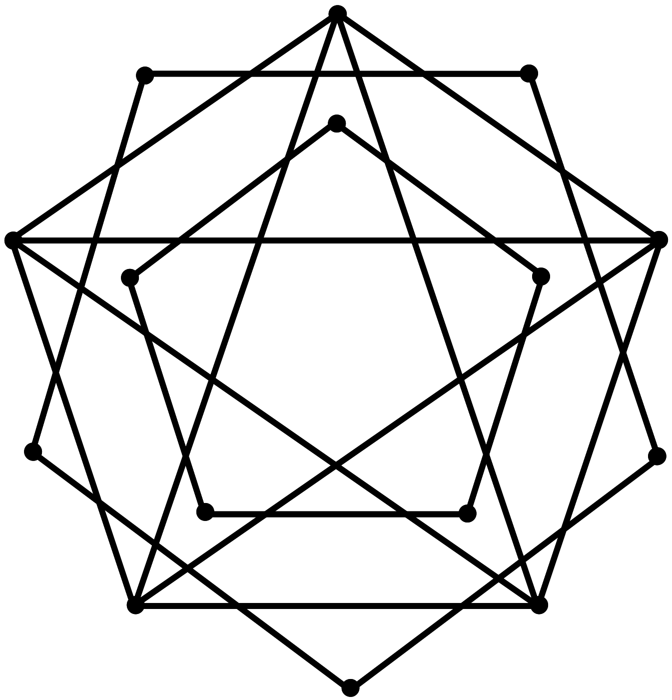

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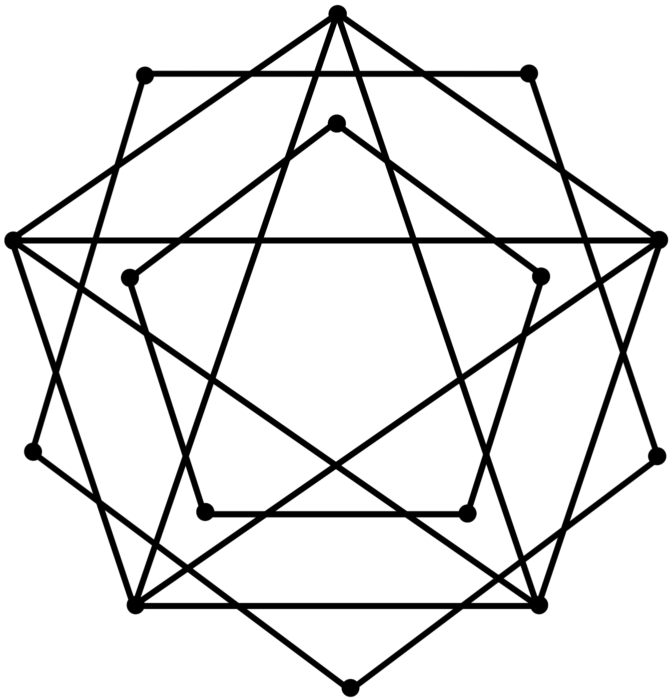

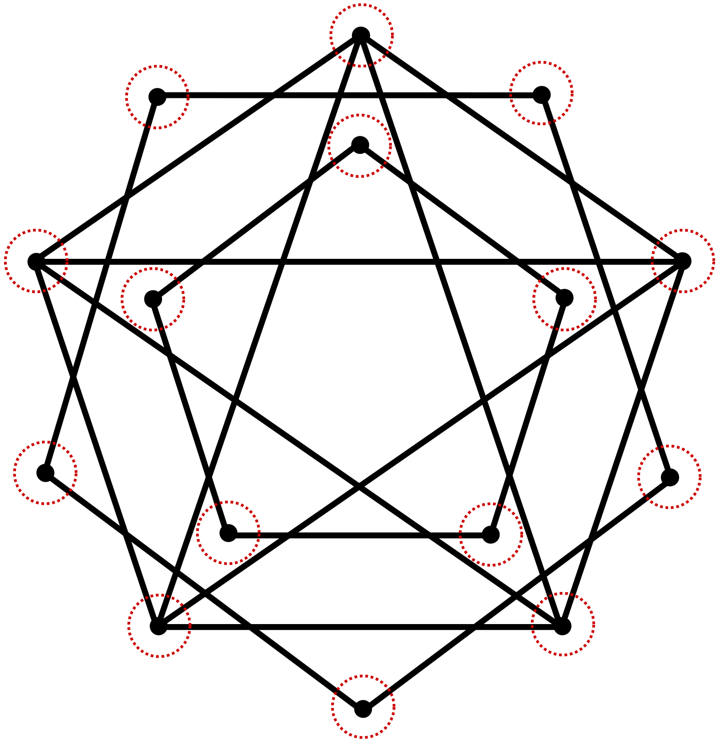

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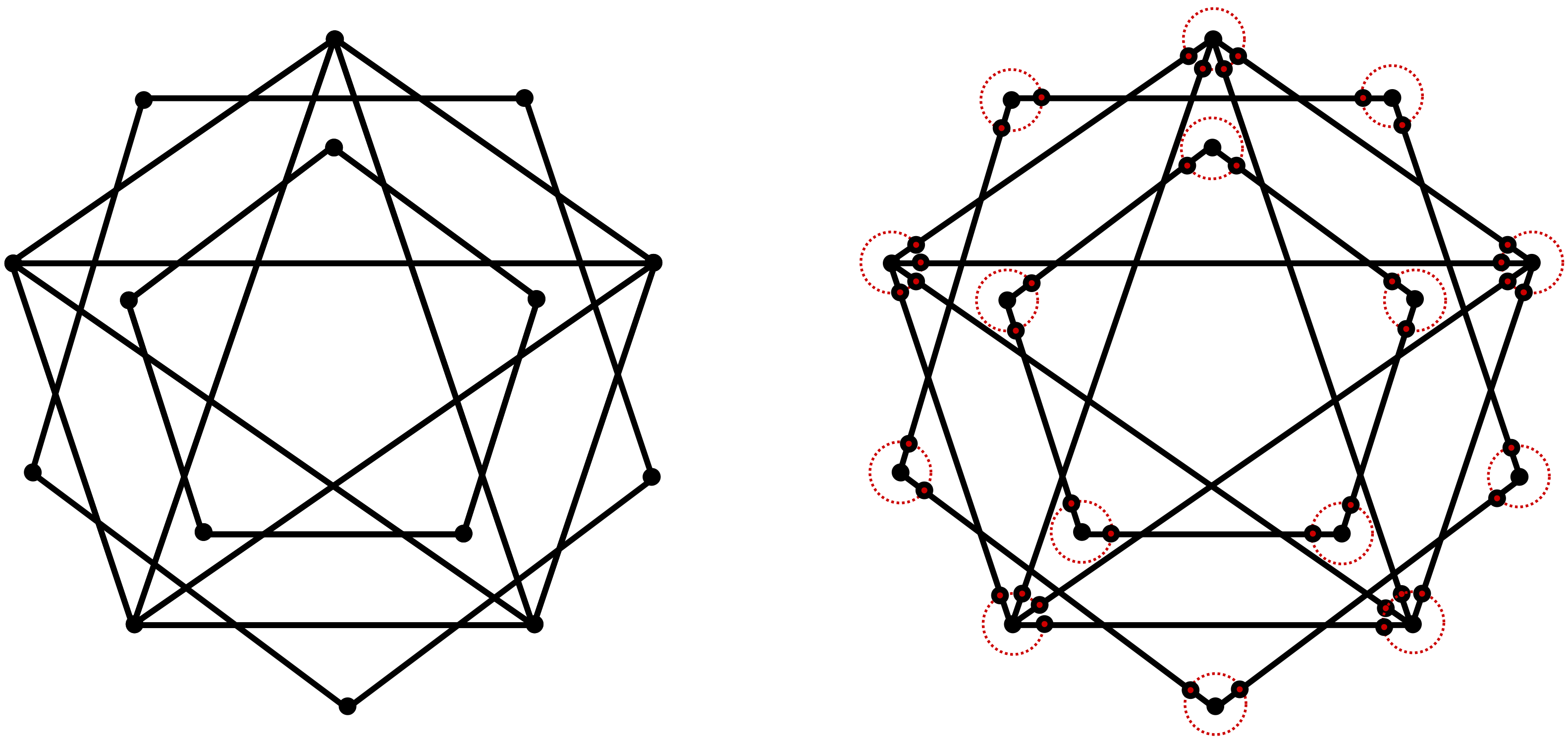

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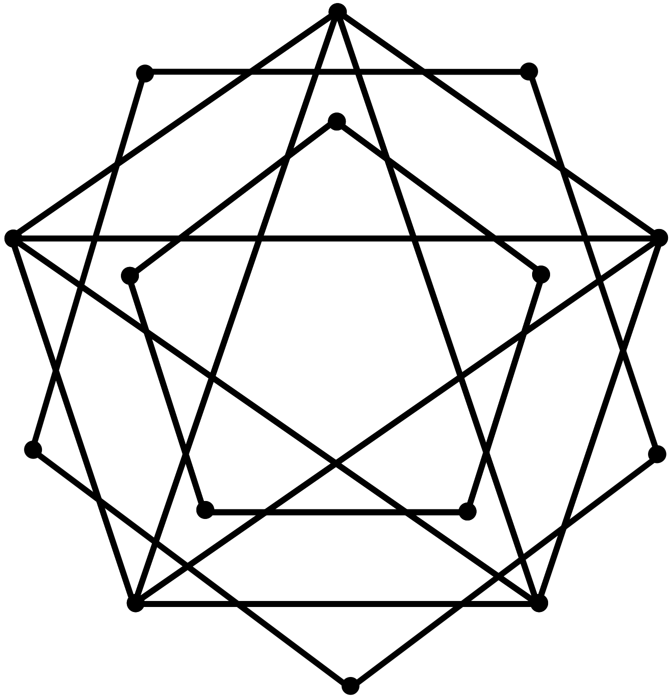

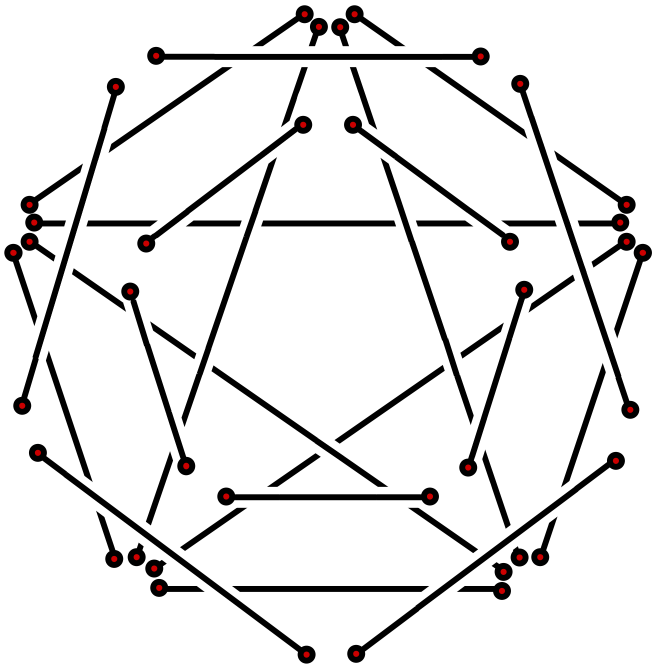

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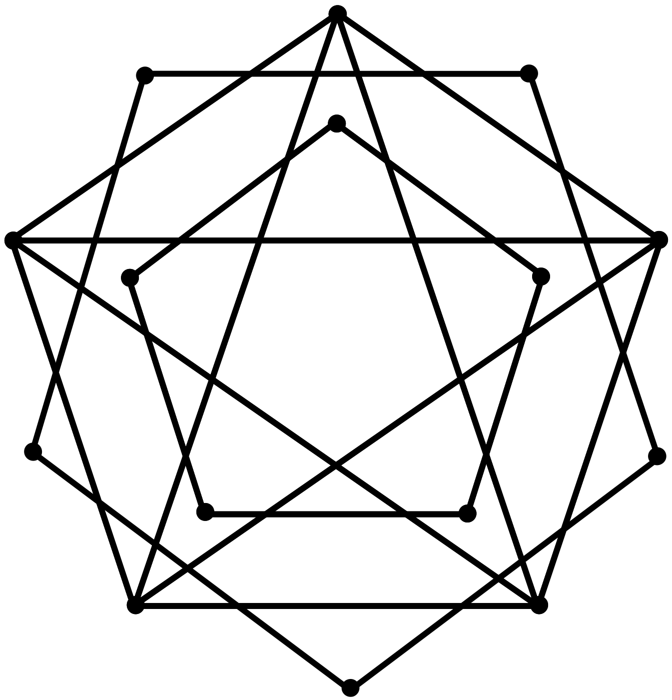
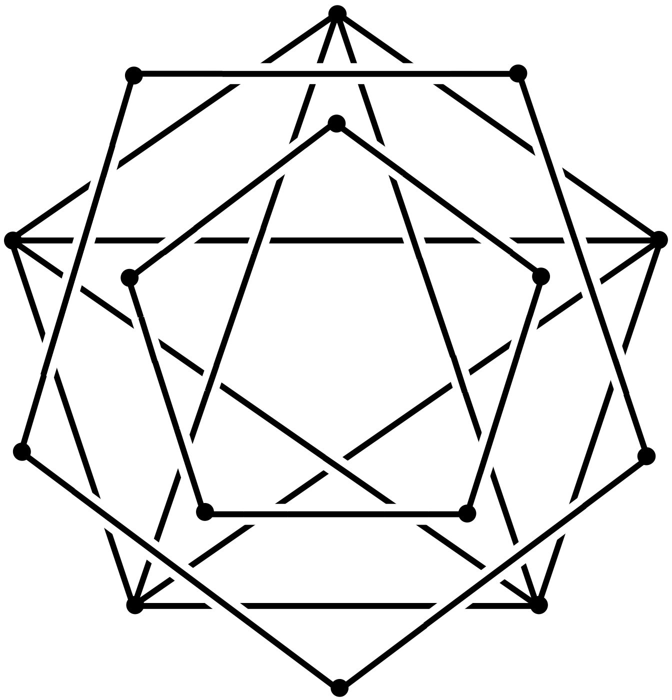
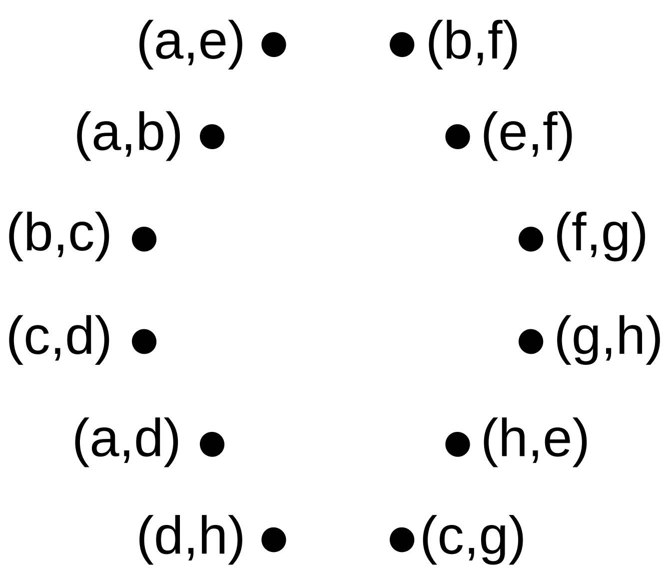


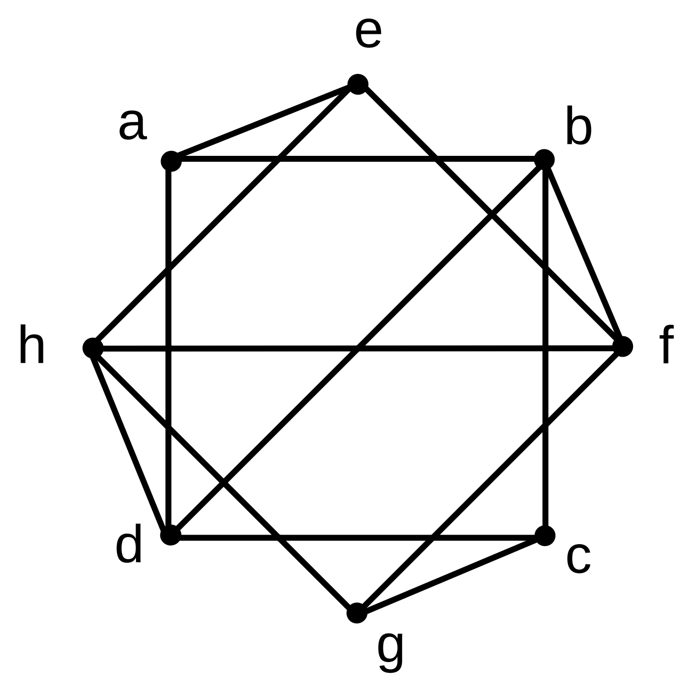


Degree-one graphs

Lemma

A drawing D of a graph G has a casing with no switches iff the crossing graph of D is bipartite.





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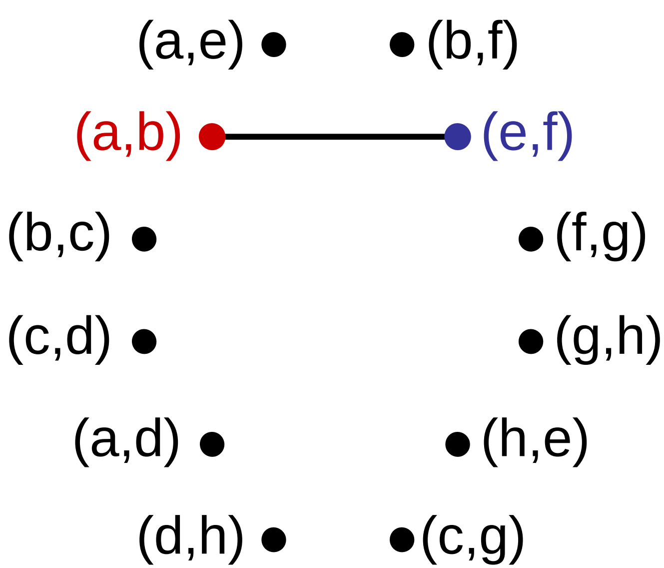

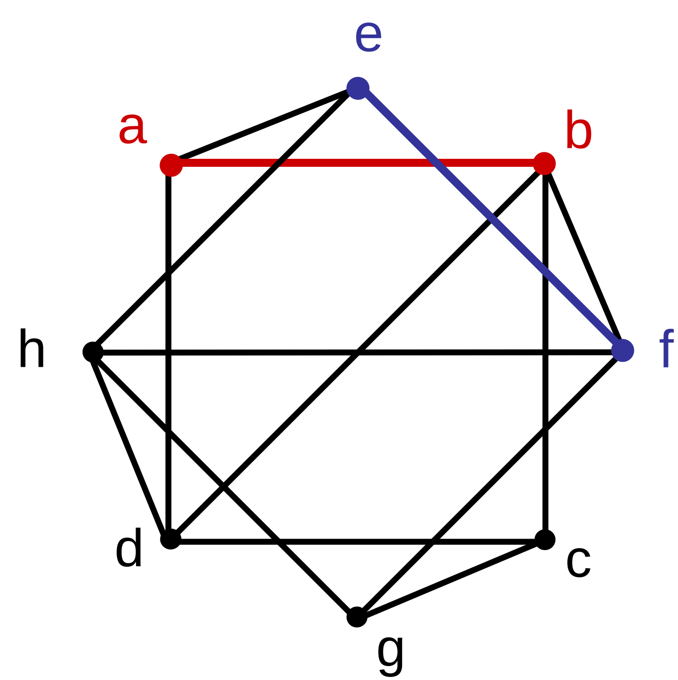

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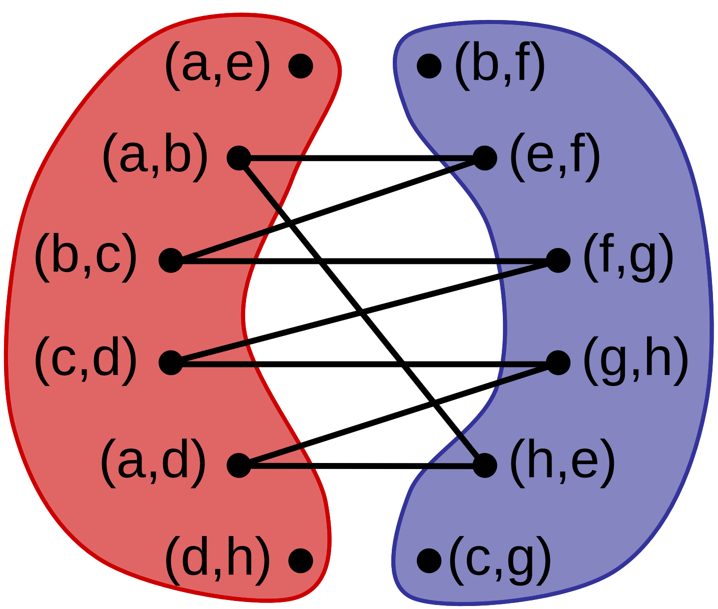

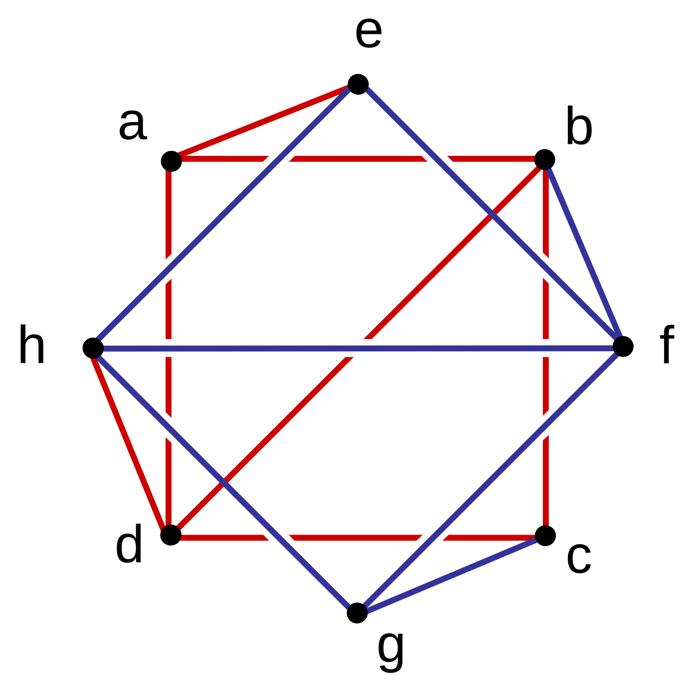

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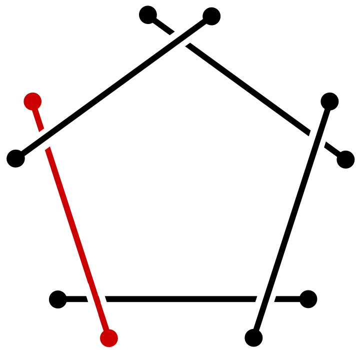
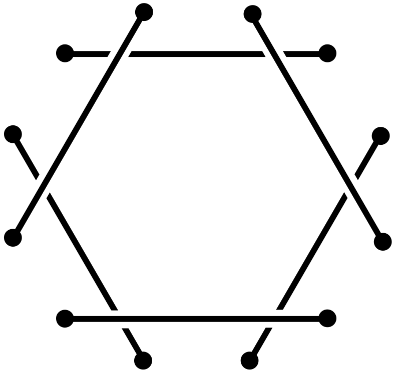


Lemma

The crossing graph of a drawing D’ of a one-degree graph G’ is bipartite iff D’ has no odd face polygons.



Degree-one graphs

Lemma

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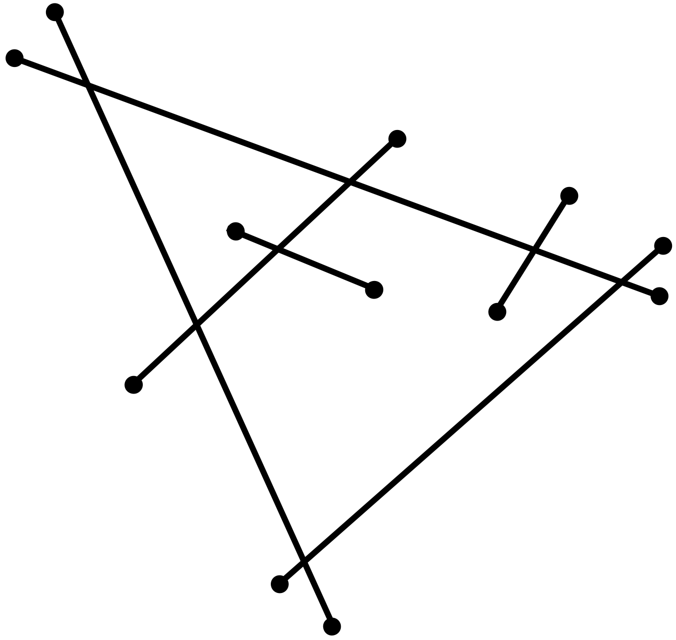

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Degree-one graphs

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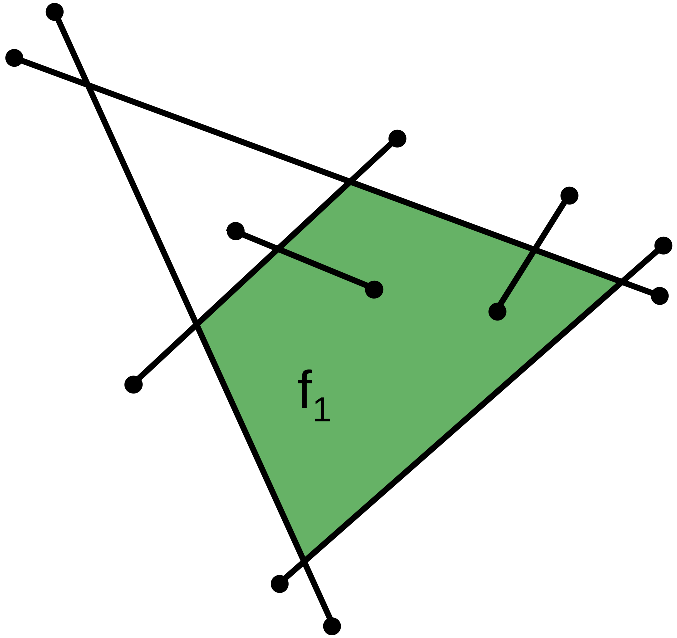
A polygon that forms the border of the closure of a face.



Degree-one graphs

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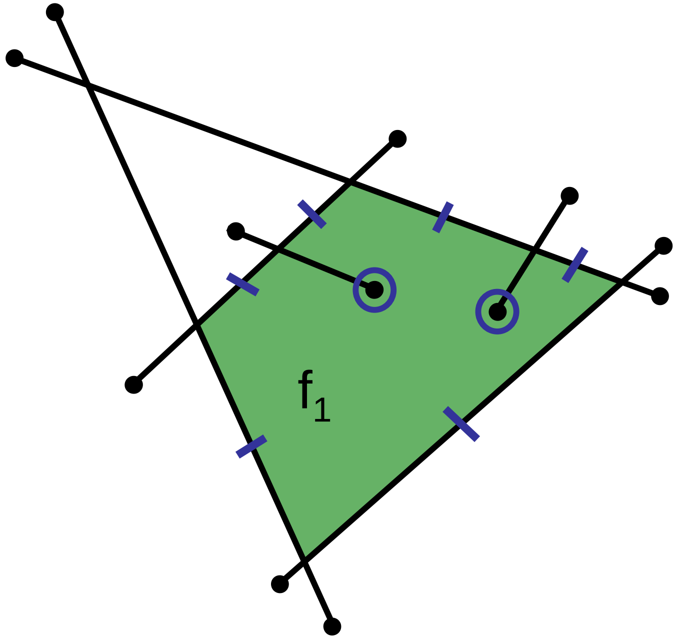
boundary segments
+
graph vertices inside



The complexity of f_1 is

6 + 2 = 8

Degree-one graphs

Lemma

A drawing D of a graph G has a casing with no switches iff the crossing graph of D is bipartite.

Lemma

The crossing graph of a drawing D’ of a one-degree graph G’ is bipartite iff D’ has no odd face polygons.

boundary segments
+
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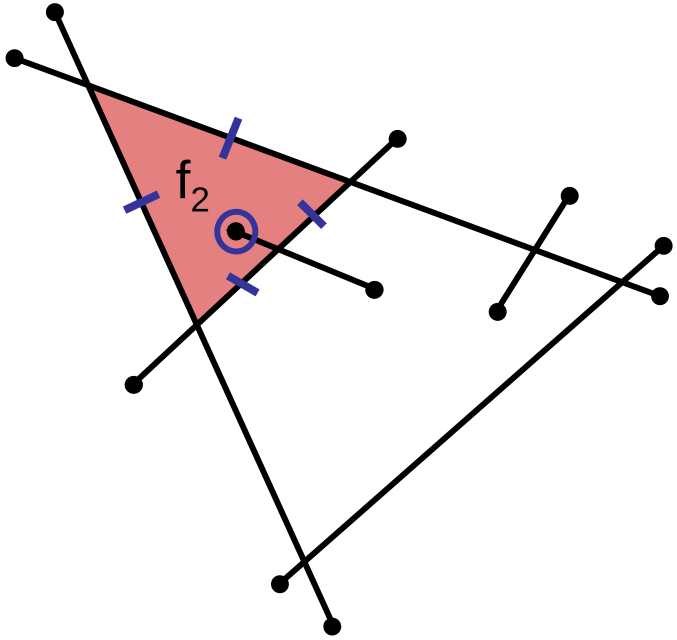

The complexity of f_2 is

4 + 1 = 5

Degree-one graphs

Lemma

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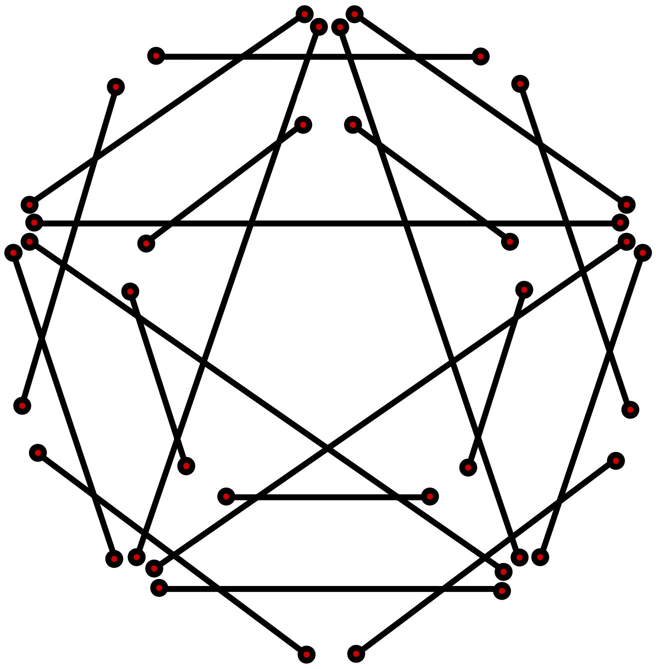
Lemma

The crossing graph of a drawing D’ of a one-degree graph G’ is bipartite iff D’ has no odd face polygons.

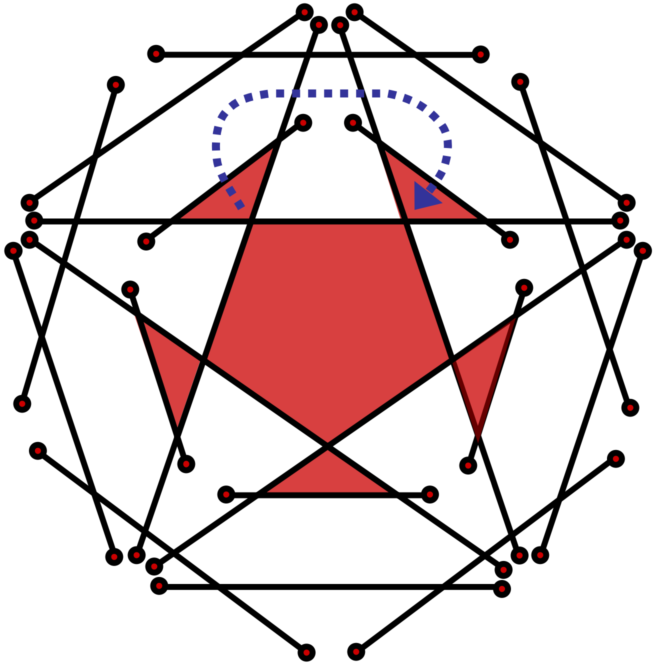


Degree-one graphs

1. Connect pairs of odd faces by cutting edges between them

⇩

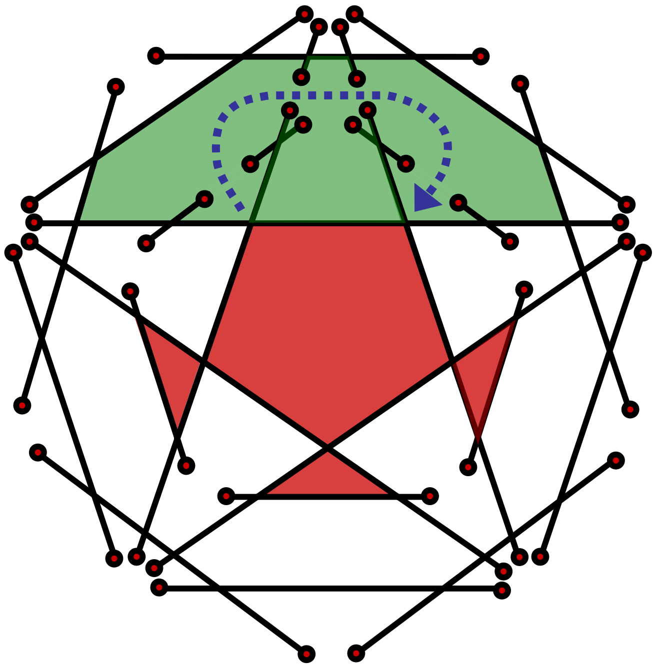
graph G* with bipartite crossing graph



Degree-one graphs

1. Connect pairs of odd faces by cutting edges between them

graph G* with bipartite crossing graph

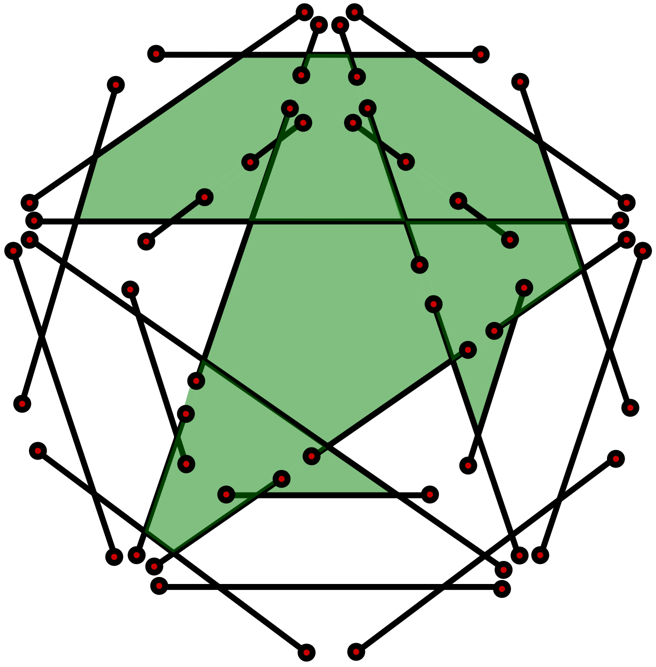


Degree-one graphs

1. Connect pairs of odd faces by cutting edges between them

⇓

graph G* with bipartite crossing graph

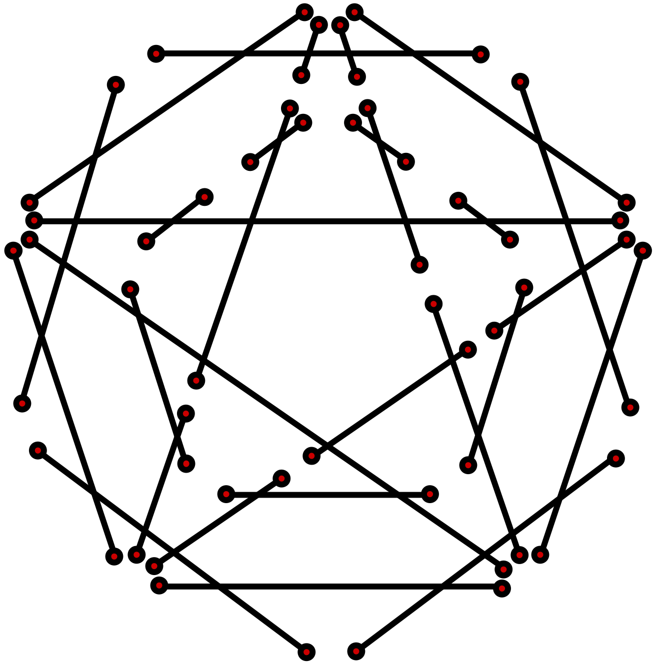


Degree-one graphs

1. Connect pairs of odd faces by cutting edges between them

graph G* with bipartite crossing graph

2. Case G*

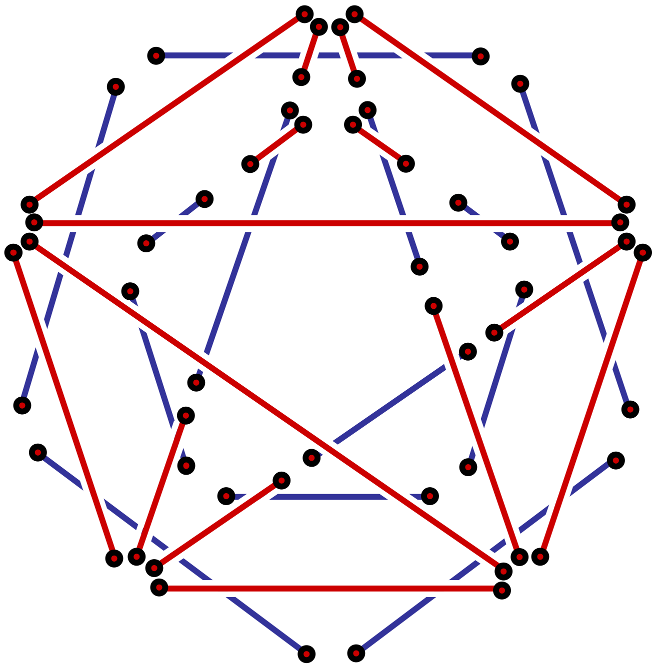


Degree-one graphs

1. Connect pairs of odd faces by cutting edges between them

graph G* with bipartite crossing graph

2. Case G*



Degree-one graphs

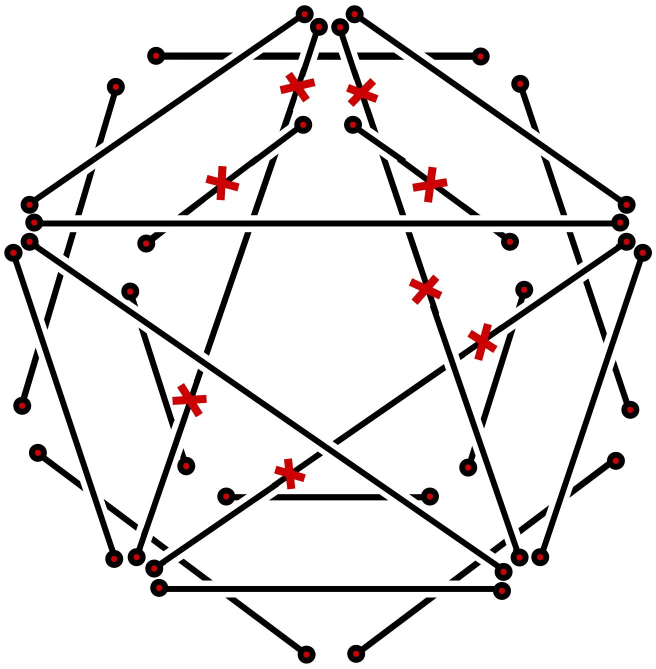
1. Connect pairs of odd faces by cutting edges between them

   graph G* with bipartite crossing graph

2. Case G*

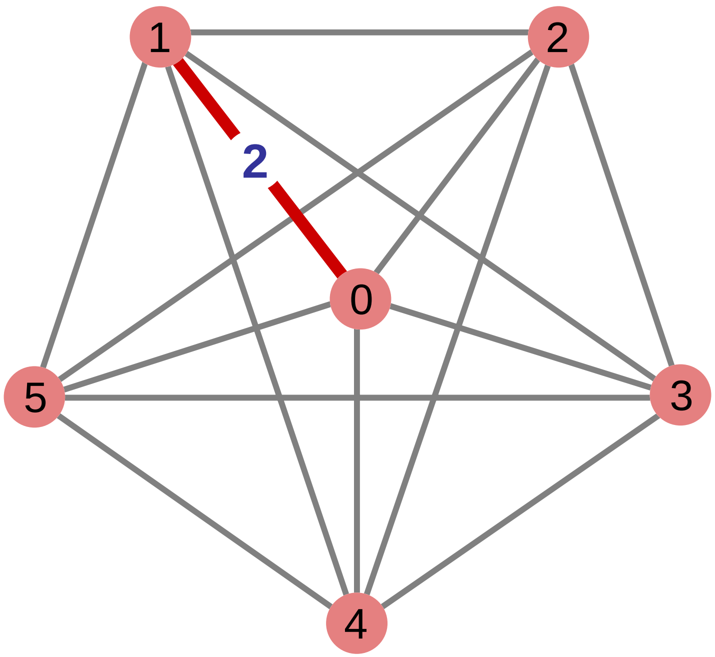
3. Merge the cut edges

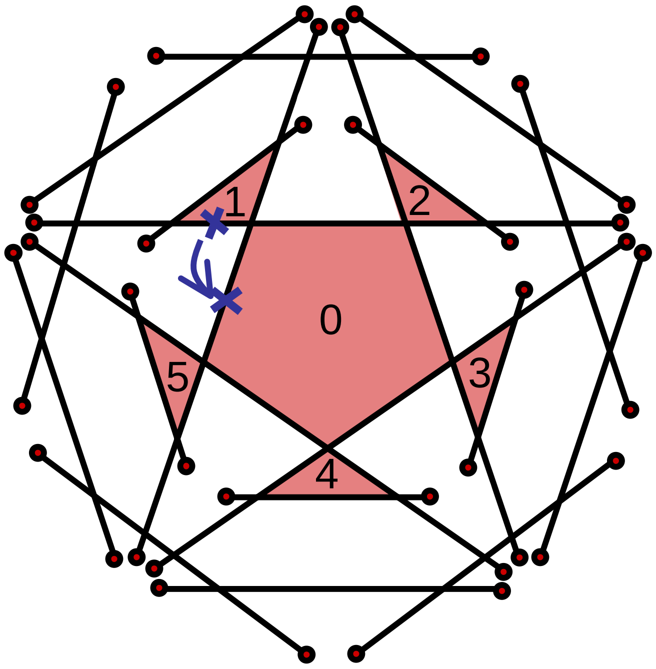
   casing for G'



Find the optimal number of cuts

Degree-one graphs



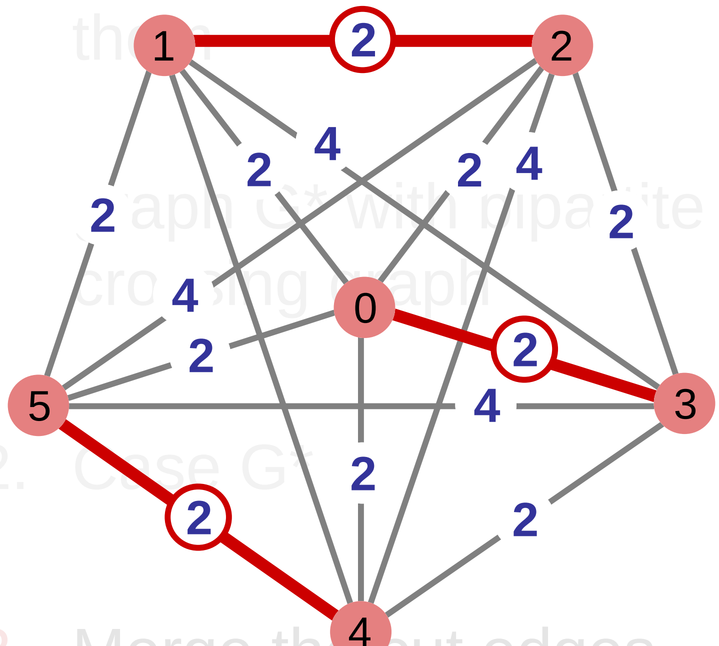


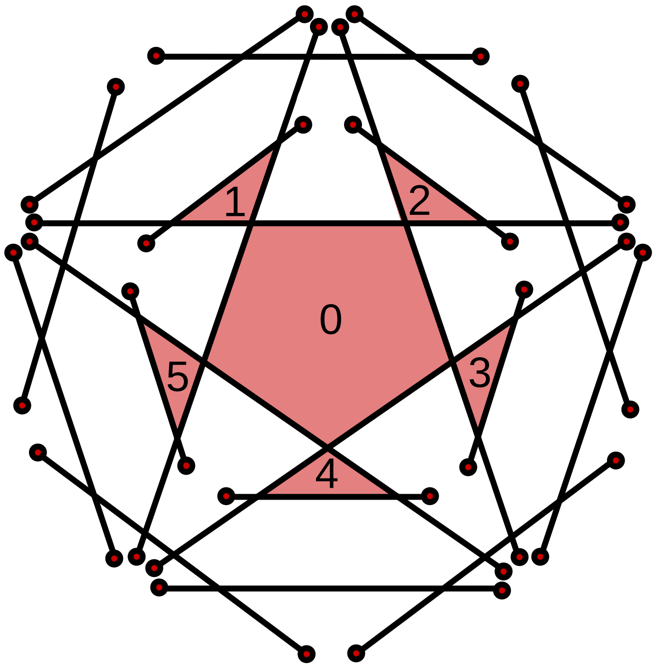
Find the optimal number of cuts

Degree-one graphs

1. Connect pairs of odd faces by cutting edges between them
2. Case G'
3. Merge the cut edges

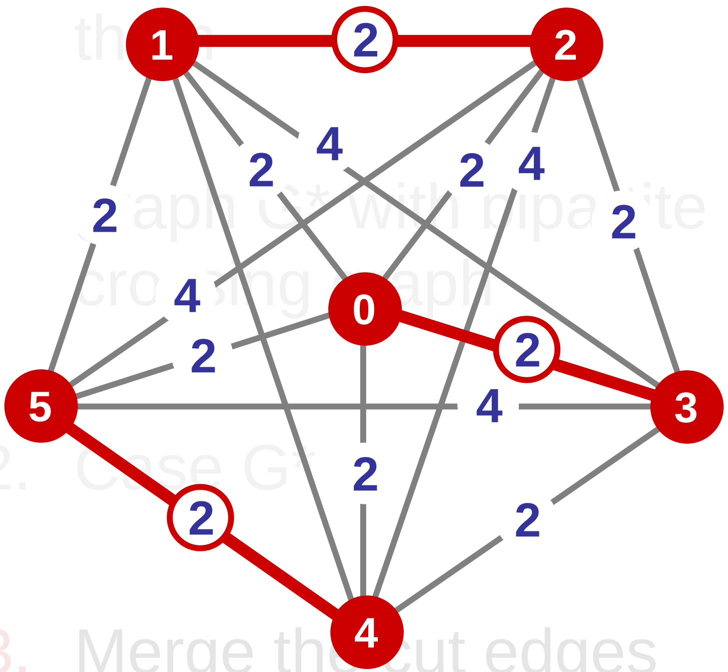
casing for G'

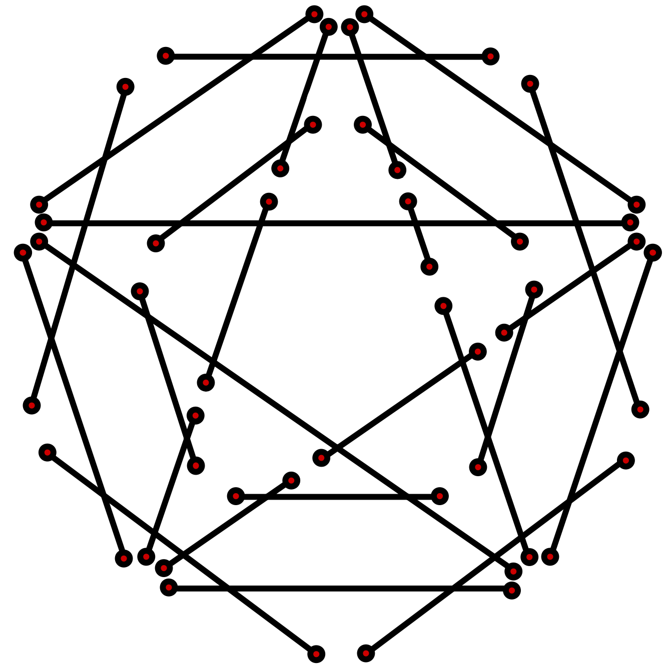




Find the optimal number of cuts

Degree-one graphs





The optimal number of cuts

Degree-one graphs

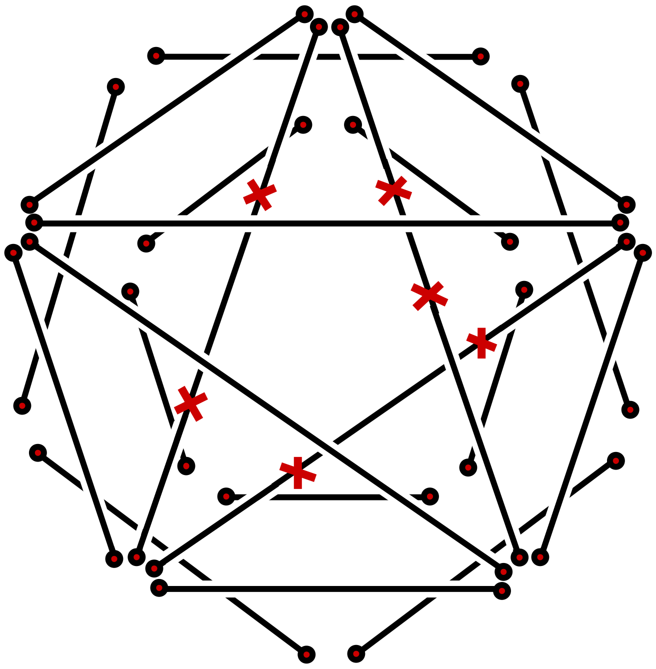
1. Connect pairs of odd faces by cutting edges between them

   graph G* with bipartite crossing graph

2. Case G*

3. Merge the cut edges

   casing for G’



The optimal casing for G’

MinTotalSwitches

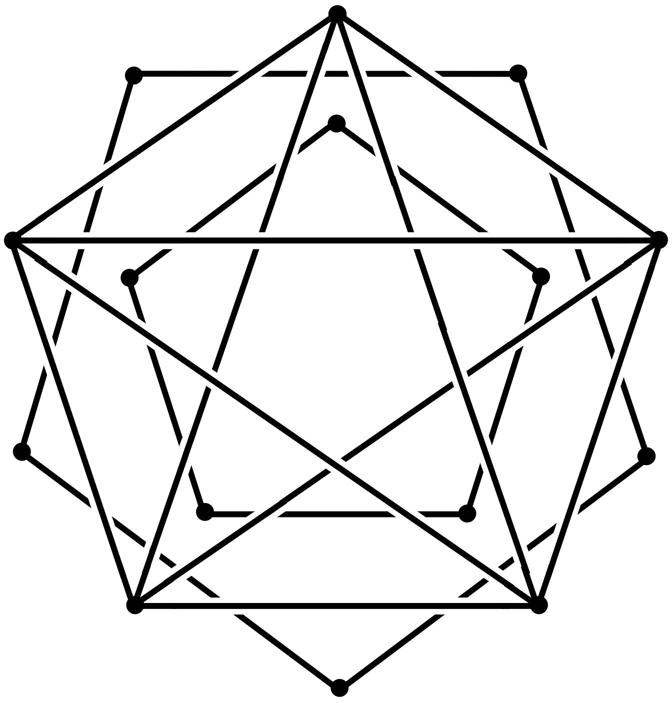
1. Connect pairs of odd faces by cutting edges between them

   graph G* with bipartite crossing graph

2. Case G*

3. Merge the cut edges

   casing for G'



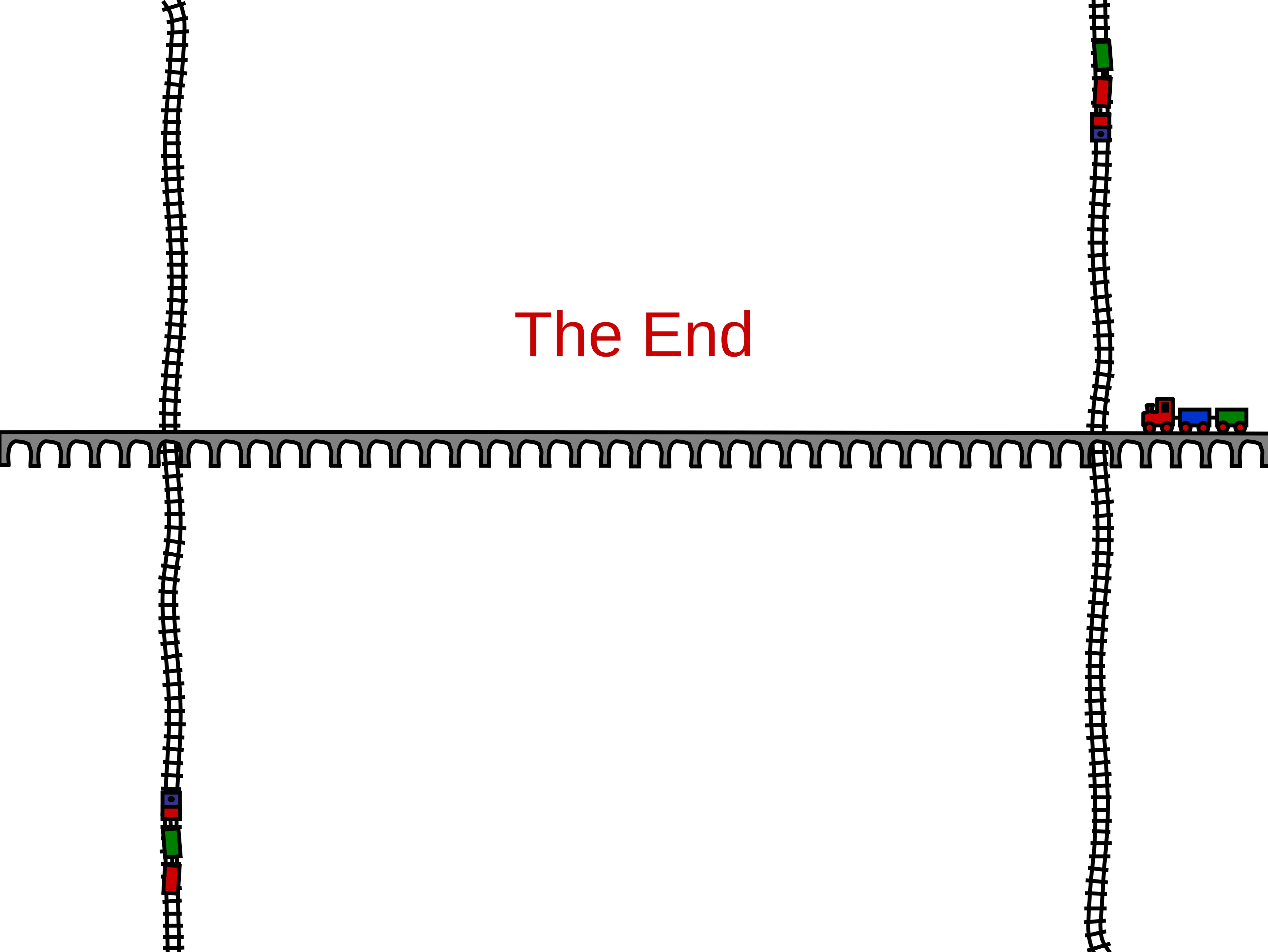
4. Optimal casing for G

Results

For a drawing D of a graph G with n vertices, m edges,
$k = O(m^2)$ crossings, $q = O(k)$ odd face polygons and
$K = O(m^3)$ total number of pairs of crossings on the same edge

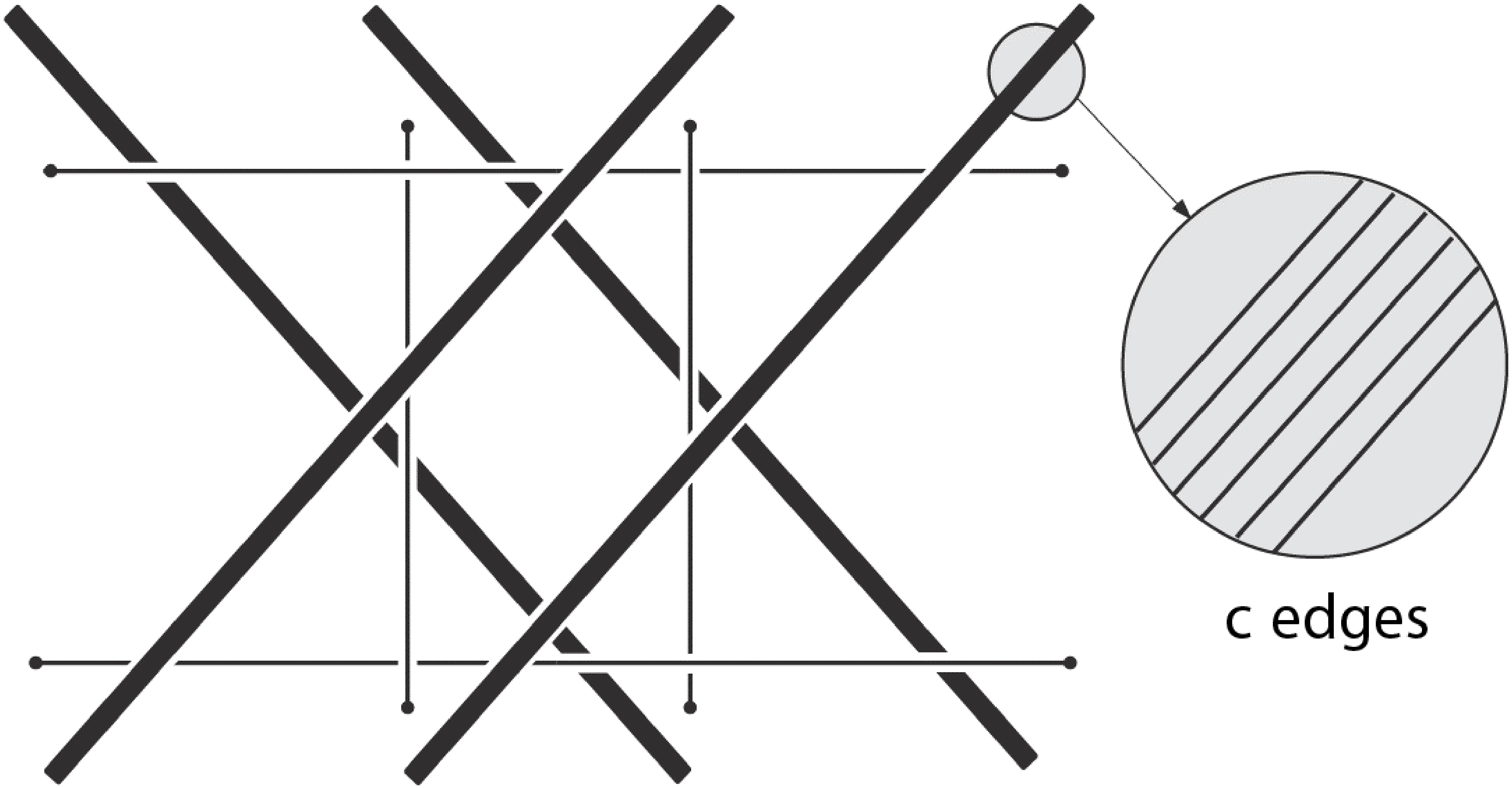
Model	Stacking	Weaving
MinTotalSwitches	*open*	$O(qk + q^{5/2} \log^{3/2} k)$
MinMaxSwitches	*open*	*open*
MinmaxTunnels	$O(m \log m + k)$ *exp.*	$O(m^4)$
MinMaxTunnelLength	$O(m \log m + k)$ *exp.*	*NP-hard*
MaxMinTunnelDistance	$O((m+k) \log m)$ *exp.*	$O((m+K) \log m)$ *exp.*

The End



Comparing models

- The realizable model is stronger than the stacking model.



Comparing models

- The realizable model is stronger than the stacking model.
- The weaving model is stronger than the realizable model.