Linear-Complexity Hexahedral Mesh Generation

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Mesh generation in theory

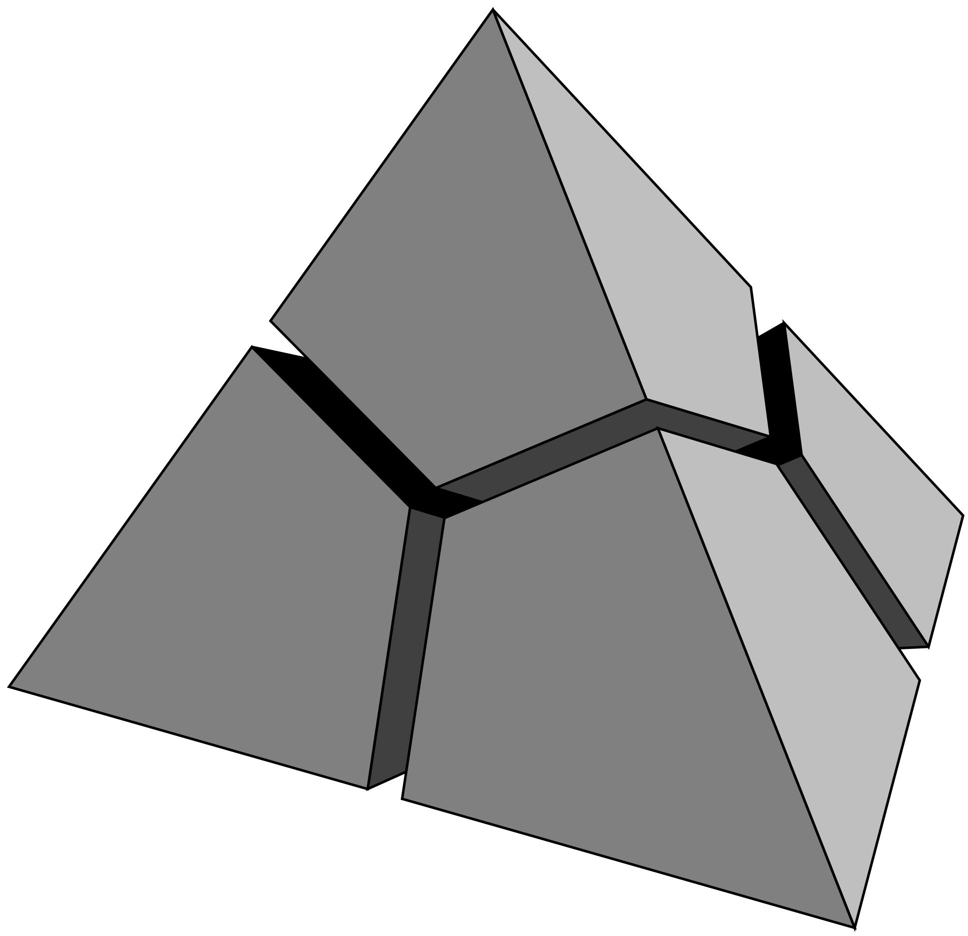
- Find triangulation
  - add diagonals to quadtree
  - incremental Delaunay refinement
  - circle packing
- Prove something
  - good element quality
  - approximation to number of elements

Mesh generation in practice

- Find preliminary mesh
  - Often quadrilaterals or hexahedra instead of triangles and tetrahedra
- Mesh improvement
  - Move Steiner points (*smoothing*)
  - Split/merge elements (*refinement*)
  - Other topological changes (*flipping*)
- Use the mesh
  - Computational fluid dynamics
  - Other finite element problems
  - Function interpolation

How to find a hexahedral mesh?

Why not just partition tetrahedra?



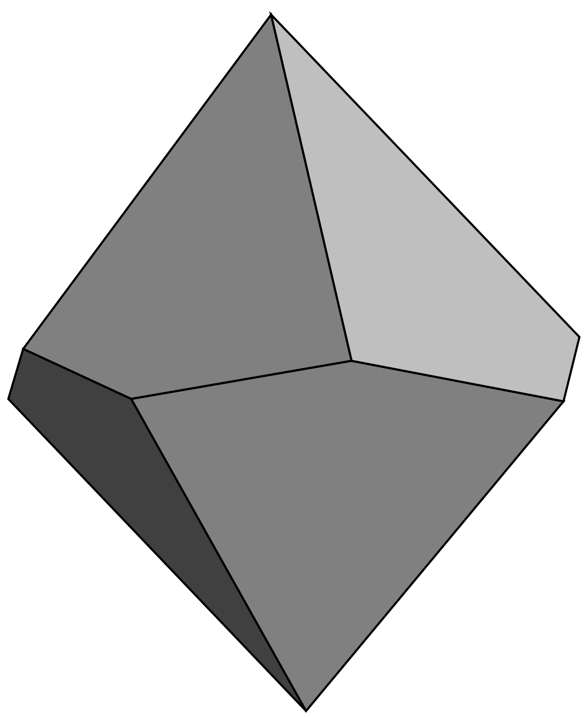
Sharp angles on boundaries can't be smoothed

Prefer to get good boundary mesh, then fill

Problem statement

Given polyhedron with quadrilateral sides

Find hexahedral mesh respecting boundary



Does this octahedron have a hexahedral mesh?

Because problem is hard, relax it:
find topological mesh (w/curved cells)
then worry about geometric embedding

Mitchell-Thurston solution

Theorem. *A polyhedron (forming a topological ball) has a topological hexahedral mesh iff it has evenly many quadrilateral sides.*

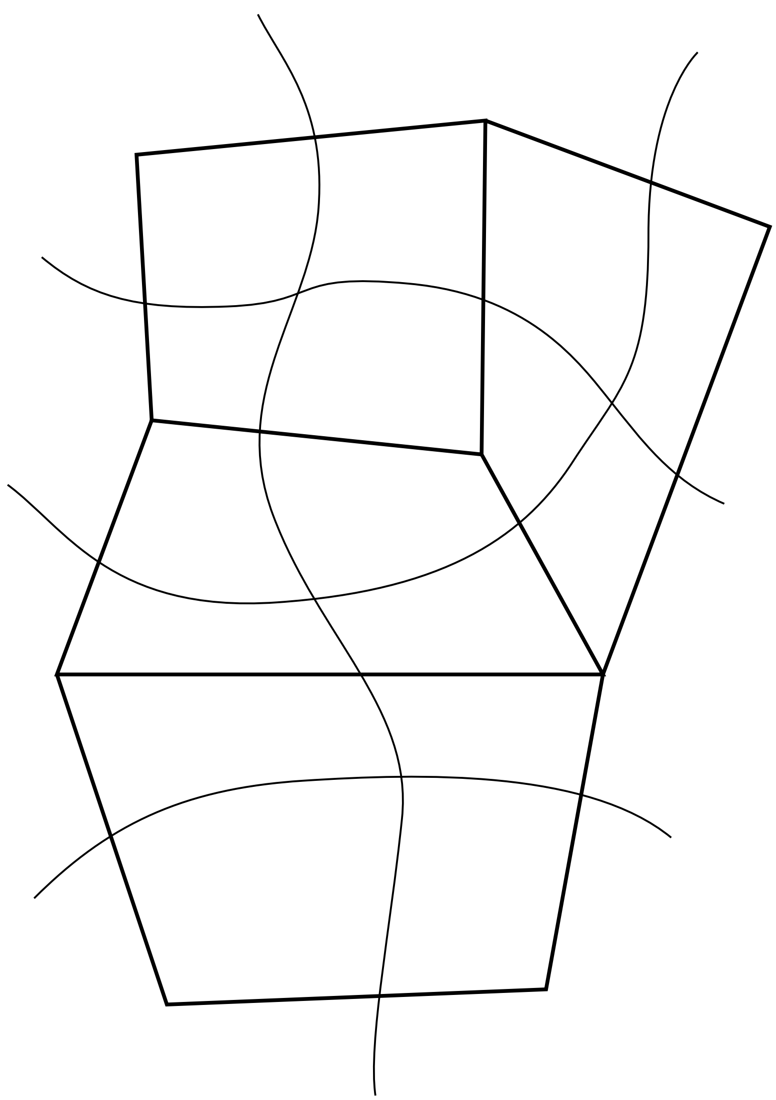
Proof:
If: by duality from existence of spanning surfaces.

Only if: every hexahedron has six sides.
Every internal boundary pairs up two sides.
So external faces must be even. □

Quadrilateral duality

Connect opposite quad edge centerpoints

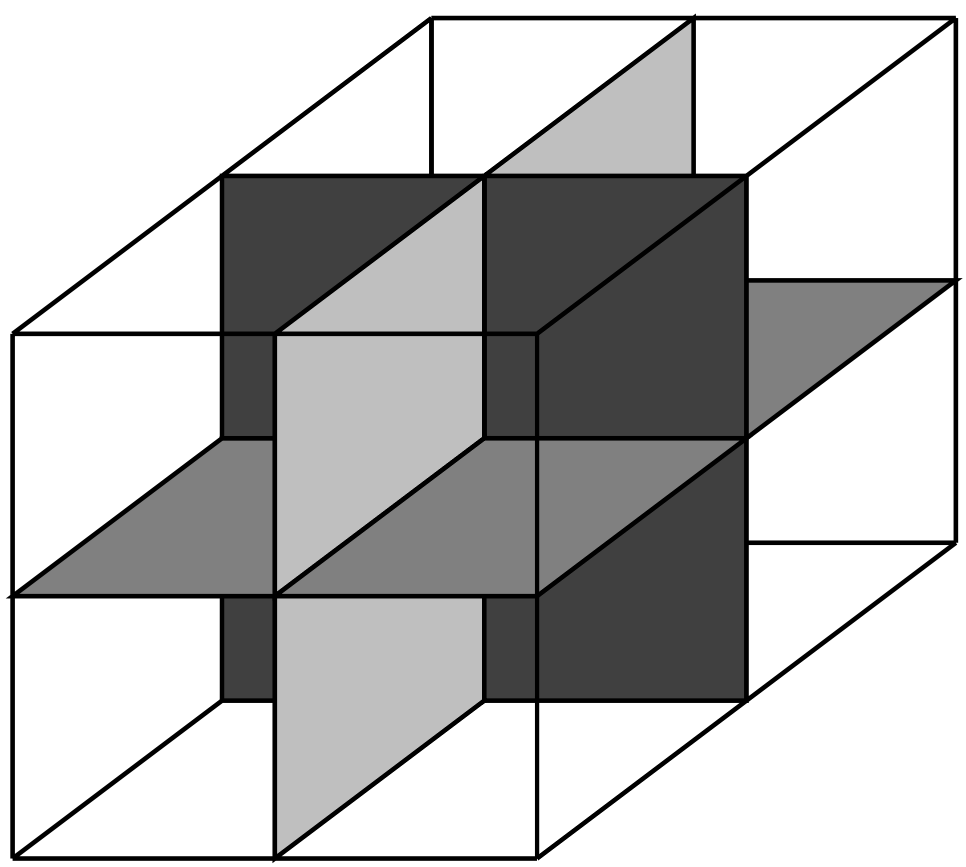
Forms arrangement of curves



Hexahedron duality

Find curve arrangements on hexahedron faces

Connect by squares meeting in hexahedron center



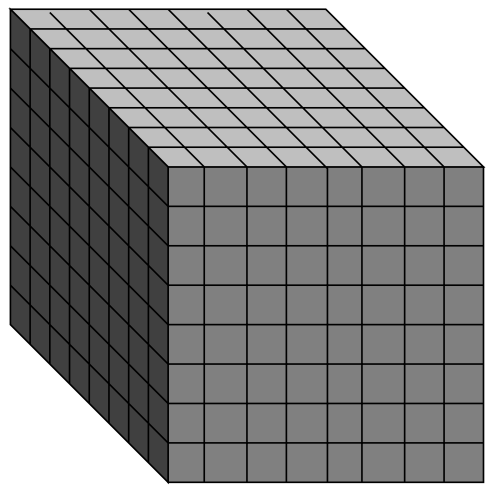
Mitchell-Thurston algorithm

- Find dual curves on boundary
- Pair curves w/odd self-intersections
- Span by surfaces
- Fix up so it has a valid dual
- Dualize to form mesh

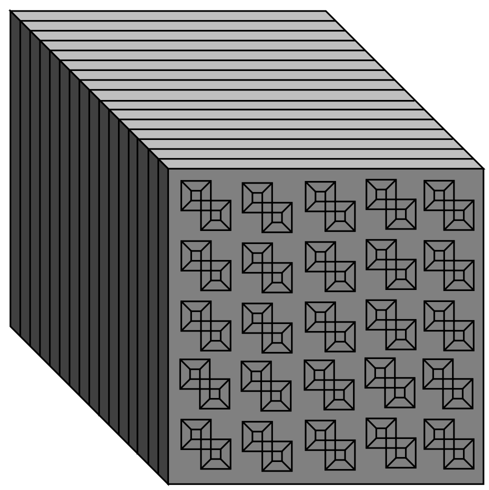
What's wrong with Mitchell-Thurston?

Produces too many hexahedra

$\Omega(n^{3/2})$:



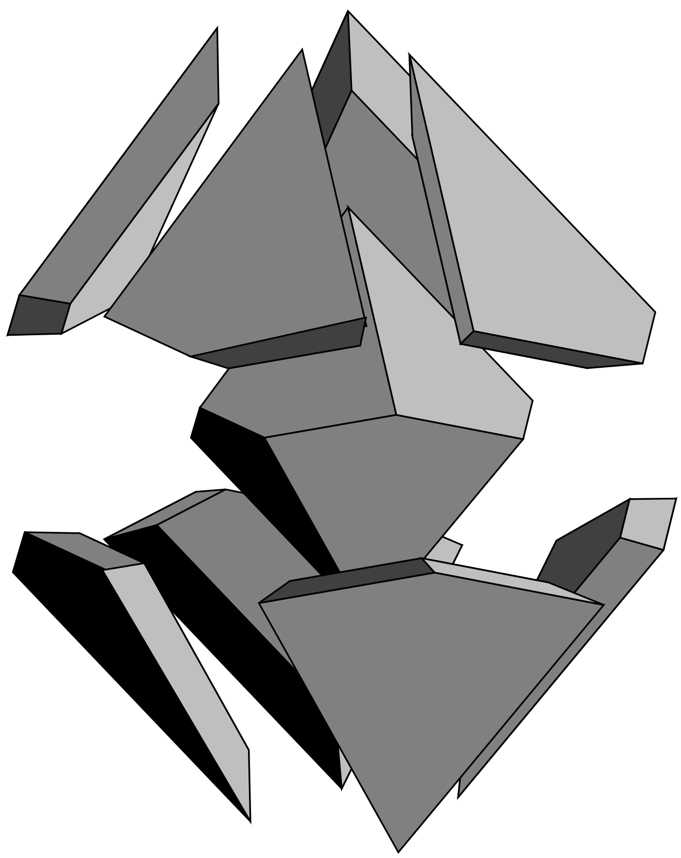
$\Omega(n^{2})$:



Doesn't result in geometric embedding.

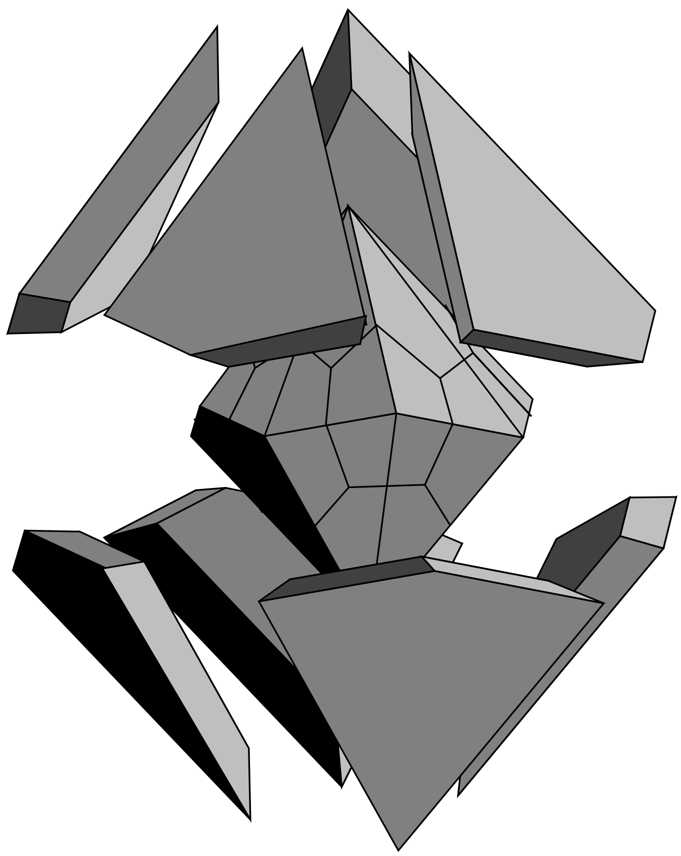
New Algorithm

I. Cover Boundary w/ Hexahedral Tiles



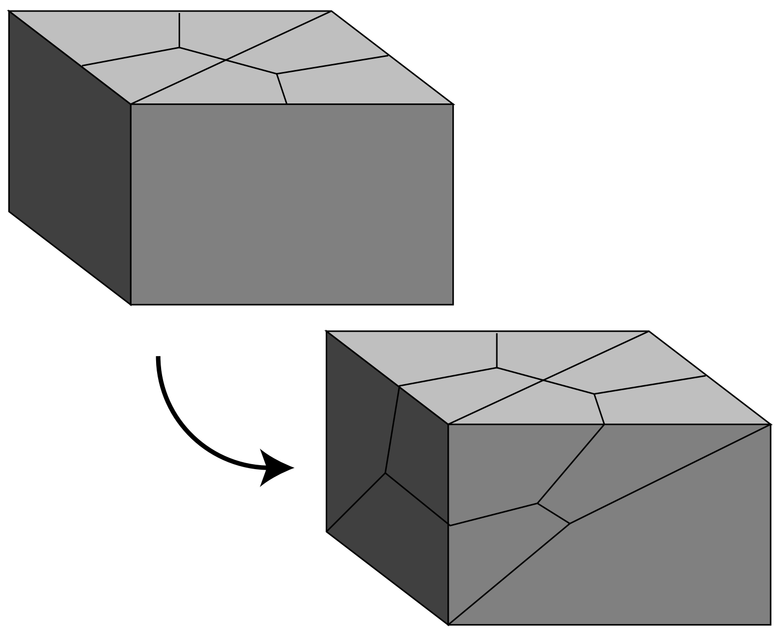
New Algorithm

II. Tetrahedralize interior and partition tetrahedra into hexahedra



New Algorithm

III. Fix up boundary tiles



- Subdivide sides, leave outside faces unchanged
- Use matching in dual graph
  to make all tiles have even # quads
- Apply Mitchell-Thurston to tiles

New results

- Complexity bound for topological hex mesh:
  If polyhedron has $2n$ quadrilateral sides,
  it has a mesh with $O(n)$ hexahedra.

- Some extensions to polyhedra
  that don't form topological balls
  (if boundary forms bipartite graph)

- Some progress in geometric embedding
  (reduction to finite case analysis)

Open Problems

- Geometric mesh (convex polyhedral hex's)?
  - More complicated boundary layer
  - May be $\Omega(n^2)$ interior hexahedra
  - Can't apply Mitchell-Thurston
- Non-simply-connected domains?
  - We have some sufficient conditions
  - Not both necessary and sufficient
- Quality of elements?
  - How good is result of algorithm?
  - How easy is it to smooth?