28. Linearly independent because the coordinate vectors
\[
\begin{bmatrix}
1 \\
0 \\
-2 \\
-1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0 \\
2
\end{bmatrix},
\begin{bmatrix}
1 \\
-2 \\
-5 \\
1
\end{bmatrix}
\] are linearly independent.

30. Linearly dependent because the coordinate vectors
\[
\begin{bmatrix}
8 \\
-12 \\
6 \\
-1
\end{bmatrix},
\begin{bmatrix}
9 \\
-6 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
1 \\
6 \\
-5 \\
1
\end{bmatrix}
\] are linearly dependent.

32. a. The coordinate vectors
\[
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
-3
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
-3
\end{bmatrix}
\] span \( \mathbb{R}^3 \). Thus these three vectors form a basis for \( \mathbb{R}^3 \) by the Invertible Matrix Theorem. Because of the isomorphism between \( \mathbb{R}^3 \) and \( \mathbb{P}_2 \), the corresponding polynomials form a basis for \( \mathbb{P}_2 \).

b. Since \( [q]_S = (-1, 1, 2) \), one may compute
\[
\begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
-3
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
-3
\end{bmatrix}
\] and \( q = 1 + 3t - 11t^2 \).

34. [M] The coordinate vectors
\[
\begin{bmatrix}
5 \\
-3 \\
4
\end{bmatrix},
\begin{bmatrix}
9 \\
1 \\
8
\end{bmatrix},
\begin{bmatrix}
6 \\
-2 \\
5
\end{bmatrix}
\] are linearly dependent. Because of the isomorphism between \( \mathbb{R}^4 \) and \( \mathbb{P}_3 \), the corresponding polynomials are linearly dependent and therefore cannot form a basis for \( \mathbb{P}_3 \).

36. [M] Row reduction of \([v_1 \, v_2 \, v_3]\) shows that there is a pivot in each column, so the columns are linearly independent and hence form a basis for the subspace \( H \) which they span.
\[
[x]_S = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}
\]

38. [M]  
\[
\begin{bmatrix}
1.30 \\
.75 \\
1.60
\end{bmatrix}
\]

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2. \[ \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 0 \end{bmatrix}; \text{ dim is } 2 \]

4. \[ \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}; \text{ dim is } 2 \]

6. \[ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \end{bmatrix}; \text{ dim is } 3 \]

8. \[ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \text{ dim is } 3 \]

10. \[ \begin{bmatrix} 1 \\ 12 \end{bmatrix}, \begin{bmatrix} 3 \\ 16 \end{bmatrix}, \begin{bmatrix} 3 \end{bmatrix}, \begin{bmatrix} 4 \end{bmatrix}, \begin{bmatrix} 16 \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, \begin{bmatrix} 2 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 2 \end{bmatrix} \]

19. a. True. See the box before Example 5.

b. False, unless the plane is through the origin. Read Example 4 carefully.

c. False. The dimension is 5. See Example 1.

d. False. \( S \) must have exactly \( n \) elements to be a basis for \( V \). See Theorem 10.

e. True. See Practice Problem 2.

20. a. False. The only subspaces of \( \mathbb{R}^3 \) are listed in Example 4. \( \mathbb{R}^2 \) is not even a subset of \( \mathbb{R}^3 \), because vectors in \( \mathbb{R}^3 \) have three coordinates. Review Example 8 in Section 4.1.

b. False. The number of free variables equals the dimension of Nul \( A \). See the box before Example 5.

c. False. Read carefully the definition before Example 1. Not being spanned by a finite set is not the same as being spanned by an infinite set. The space \( \mathbb{R}^2 \) is finite-dimensional, yet it is spanned by the infinite set \( S \) of all vectors of the form \((x,y)\), where \( x \) and \( y \) are integers. (Of course, the two vectors \((1,0)\) and \((0,1)\) in \( S \) by themselves span \( \mathbb{R}^2 \).)

d. False. \( S \) must have exactly \( n \) elements to be a basis of \( V \). See the Basis Theorem.

e. True. See Example 4.

22. Obviously, none of the Laguerre polynomials is a linear combination of the Laguerre polynomials of lower degree. By Theorem 4 (Section 4.3), the set of polynomials is linearly independent. Since this set contains four vectors, and \( \mathbb{P}_3 \) is four-dimensional, the set is a basis of \( \mathbb{P}_3 \), by the Basis Theorem.

24. \( [p]_S = (6, 3, -2) \)

26. If \( \dim V = 0 \), the statement is obvious. Otherwise, \( H \) contains a basis, consisting of \( n \) linearly independent vectors. By the Basis Theorem applied to \( V \), the vectors form a basis for \( V \).

28. The space \( C(\mathbb{R}) \) contains the space \( \mathbb{P} \) as a subspace. If \( C(\mathbb{R}) \) were finite-dimensional, \( \mathbb{P} \) would be finite-dimensional, too, by Theorem 11. This is not true, by Exercise 27, so \( C(\mathbb{R}) \) is infinite-dimensional.

30. a. False. This is not Theorem 9. If \( x \in V \) is nonzero, the set \( \{0, x, 2x, \ldots, (p-1)x\} \) is linearly dependent, no matter what the dimension of \( V \).

b. True. If \( \dim V \) were less than or equal to \( p \), \( V \) would have a basis of not more than \( p \) elements. Such a set
would span $V$. Since this is not the case, dim $V$ must be greater than $p$.

c. False. Counterexample: Take any nonzero vector $v$, and consider the set \{v, 2v, 3v, \ldots, (p-1)v\}.

32. Let \{u_1, \ldots, u_p\} be a basis for $H$. Then \{T(u_1), \ldots, T(u_p)\} spans $T(H)$, as is easily seen. Further, since $T$ is one-to-one, Exercise 32 in Section 4.3 shows that \{T(u_1), \ldots, T(u_p)\} is linearly independent. So this set of images is a basis for $T(H)$. So dim $H = p$ and dim $T(H) = p$.

33. [M] a. \{v_1, v_2, v_3, e_2, e_3\}

b. The first $k$ columns of $A$ are pivot columns because, by assumption, the original $k$ vectors are linearly independent. Col $A = \mathbb{R}^n$, because the columns of $A$ include all the columns of the identity matrix.

34. [M] The $B$-coordinate vectors of the vectors in $C$ are the columns of the matrix

\[
P = \begin{bmatrix}
1 & 0 & -1 & 0 & 1 & 0 & -1 \\
1 & 0 & -3 & 0 & 5 & 0 & 18 \\
2 & 0 & -8 & 0 & 18 & 0 \\
4 & 0 & -20 & 0 & 8 & 0 & 48 \\
8 & 0 & -16 & 0 & 16 & 0 & 32 \\
\end{bmatrix}
\]

a. This problem is an [M] exercise because it involves a large matrix. However, one should always think about a problem before rushing to use a matrix program. Actually, neither part of this exercise requires a matrix program. Simply observe that the matrix $P$ is invertible because it is triangular with nonzero entries on the diagonal. So the columns of $P$ are linearly independent. Because the coordinate mapping is an isomorphism, the vectors in $C$ are linearly independent.

b. dim $H = 7$, because $B$ is a basis for $H$ with 7 elements. Since $C$ is linearly independent, and the vectors in $C$ lie in $H$ (because of the trig identities), $C$ is a basis for $H$, by the Basis Theorem. (Another argument is to use the fact that the $B$-coordinate vectors of the vectors in $C$ span $\mathbb{R}^7$, so the vectors in $C$ span $H$. But you must distinguish between vectors in $\mathbb{R}^7$ and vectors in $H$.)

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2. rank $A = 3$; dim Null $A = 2$;

\[
\begin{bmatrix}
1 & 4 & 2 \\
2 & 6 & 3 \\
3 & 3 & 3 \\
3 & 0 & 0 \\
\end{bmatrix}
\]

Basis for Col $A$: \{1, 3, 4, -1, 2\}, (0, 0, 1, -1, 1), (0, 0, 0, 0, -5)

Basis for Row $A$: (1, 3, 4, -1, 2), (0, 0, 1, -1, 1), (0, 0, 0, 0, -5)

4. rank $A = 5$; dim Null $A = 1$;

\[
\begin{bmatrix}
-3 & 0 \\
1 & 0 \\
0 & 0 \\
-2 & 1 \\
-3 & 0 \\
\end{bmatrix}
\]

Basis for Col $A$: \{1, 1, -1, -1, 5\}, (0, 0, 1, -1, -3, -1), (0, 0, 1, -13, -1), (0, 0, 0, 0, 0, 1)

Basis for Row $A$: \{1, 1, -1, -1, 5\}, (0, 0, 1, -1, -3, -1), (0, 0, 0, 0, 0, 1)

6. 3, 2, 2

8. 4. It is impossible for Col $A$ to be $\mathbb{R}^4$ because the vectors in Col $A$ have 6 entries. Col $A$ is a four-dimensional subspace of $\mathbb{R}^6$.

10. 2

12. 2

14. 4, 4. If $A$ is $5 \times 4$, its rows are in $\mathbb{R}^4$ and there can be at most four linearly independent vectors in such a set. If $A$ is $4 \times 5$, it cannot have more than four linearly independent rows because there are only four rows.

16. 0

17. a. True. The row vectors in $A$ are identified with the columns of $A^T$. See the paragraph before Example 1.

b. False. See the warning after Example 2.

c. True. See the Rank Theorem.

d. False. See the Rank Theorem. The sum of the two dimensions equals the number of columns in $A$.

e. True. See the Numerical Note before the Practice Problems.

18. a. False. Review the warning after the proof of Theorem 6 in Section 4.3.

b. False. See the warning after Example 2. For instance, a row interchange usually changes dependence relations among the rows.

c. True. See the remark in the proof of the Rank Theorem.

d. True. This fact was noted in the paragraph before Example 4. It also follows from the fact that the rows of a matrix—say, $A^T$—are the columns of its transpose, and $A^{TT} = A$.

e. True. See Theorem 13.

20. No. The presence of two free variables indicates that the null space of the coefficient matrix $A$ is two-dimensional. Since there are eight unknowns, $A$ has eight columns and