$x_1 = Ax_0 + Bu_0 = Bu_0$
$x_2 = Ax_1 + Bu_1 = ABu_0 + Bu_1$
$x_3 = Ax_2 + Bu_2 = (ABu_0 + Bu_1) + Bu_2$
$x_4 = Ax_3 + Bu_3 = (ABu_0 + ABu_1 + Bu_2) + Bu_3$

$x_4 = Ax_3 + Bu_3$

$A = A^2 Bu_0 + ABu_1 + ABu_2 + Bu_3$

$\begin{bmatrix}
B & AB & A^2 B & A^3 B
\end{bmatrix}
\begin{bmatrix}
u_3 \\
u_2 \\
u_1 \\
u_0
\end{bmatrix}
= Mu$, where $u$ is in $\mathbb{R}^4$.

b. If $(A, B)$ is controllable, then the controllability matrix $M$ has rank 4, with a pivot in each row, and the columns of $M$ span $\mathbb{R}^4$. Therefore, for any vector in $v$ in $\mathbb{R}^4$, there is a vector $u$ in $\mathbb{R}^n$ such that $v = Mu$. However, from part (a) we know that $x_4 = Mu$ when $u$ is partitioned into a control sequence $u_3, \ldots, u_1$. This particular control sequence makes $x_4 = v$.

20. $\begin{bmatrix}
B & AB & A^2 B
\end{bmatrix}$

rank less than 3, so the pair $(A, B)$ is not controllable.

22. $[M]$ rank $\begin{bmatrix}
B & AB & A^2 B & A^3 B
\end{bmatrix} = 4$.
The pair $(A, B)$ is controllable.

Chapter 5

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2. Yes 4. Yes, $\lambda = 3$ 6. No

8. Yes, $\begin{bmatrix}
0 \\
3 \\
2
\end{bmatrix}$

10. $\begin{bmatrix}
-2 \\
1 \\
1
\end{bmatrix}$

12. $\lambda = 3; \begin{bmatrix}
-1 \\
1
\end{bmatrix}; \lambda = 7; \begin{bmatrix}
1 \\
3
\end{bmatrix}$

14. $\begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}$

16. $\begin{bmatrix}
1 \\
0 \\
0 \\
1
\end{bmatrix}$

18. 5, 0, 3

20. $\lambda = 0$. Eigenvectors for $\lambda = 0$ have entries that produce linear dependence relations among the columns of $A$. Any nonzero vector in $\mathbb{R}^3$ whose entries sum to 0 will work. Find any two such vectors that are not multiples of each other; for example, $\begin{bmatrix}
1 \\
1 \\
-2
\end{bmatrix}$ and $\begin{bmatrix}
1 \\
-1 \\
0
\end{bmatrix}$

21. a. False. The equation $Ax = \lambda x$ must have a nontrivial solution.

b. True. See the paragraph after Example 5.

c. True. See the discussion of equation (3).

d. True. See Example 2 and the paragraph preceding it. Also, see the Numerical Note.

e. False. See the warning after Example 3.

22. a. False. The vector $x$ in $Ax = \lambda x$ must be nonzero.

b. False. See Example 4 for a two-dimensional eigenspace, which contains two linearly independent eigenvectors corresponding to the same eigenvalue. The statement given is not at all the same as Theorem 2. In fact, it is the converse of Theorem 2 (for the case $r = 2$).

c. True. See the paragraph after Example 1.

d. False. Theorem 1 concerns a triangular matrix. See Examples 3 and 4 for counterexamples.

e. True. See the paragraph following Example 3. The eigenspace of $A$ corresponding to $\lambda$ is the null space of the matrix $A - \lambda I$.

24. Any triangular matrix with the same number in both diagonal entries, such as $\begin{bmatrix}
4 & 5 \\
0 & 4
\end{bmatrix}$

26. If $Ax = \lambda x$ for some $x \neq 0$, then $A^2 x = A(Ax) = A(\lambda x) = \lambda A x = \lambda^2 x$.

However, $A^2 x = 0$ because $A^2 = 0$. Therefore, $0 = \lambda^2 x$.
Since $x \neq 0$, we conclude that $\lambda$ must be zero. Thus each eigenvalue of $A$ is zero.

28. If $A$ is lower triangular, then $A^T$ is upper triangular and has the same diagonal entries as $A$. Hence, by the part of Theorem 1 already proved in the text, these diagonal entries are eigenvalues of $A^T$. By Exercise 27, they are also eigenvalues of $A$.

30. By Exercise 29 applied to $A^T$ in place of $A$, we conclude that $\lambda$ is an eigenvalue of $A^T$. By Exercise 27, $\lambda$ is an eigenvalue of $A$.

32. Suppose $T$ rotates points about some line $L$ that passes through the origin in $\mathbb{R}^3$. That line consists of all multiples of some nonzero vector $v$. The points on this line do not move under the action of $T$. So $T(v) = v$. If $A$ is the standard matrix of $T$, then $Av = v$. Thus $v$ is an eigenvector of $A$ corresponding to the eigenvalue 1. The eigenspace is Span $\{v\}$.

If the rotation happens to be half of a full rotation, that is, through an angle of 180 degrees, let $P$ be a plane through the origin that is perpendicular to the line $L$. Each point $p$ in this plane rotates into $-p$. That is, each point in $P$ is an eigenvector of $A$ corresponding to the eigenvalue $-1$.

34. You could try to write $x_0$ as a linear combination of eigenvectors, $v_1, ..., v_p$, of $A$. If $\lambda_1, ..., \lambda_p$ are corresponding eigenvalues, and if $x_0 = c_1 v_1 + ... + c_p v_p$, then you could define $x_k = c_1 \lambda_1^k v_1 + ... + c_p \lambda_p^k v_p$.

In this case, for $k = 0, 1, 2, ...$.
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2. \[
\begin{bmatrix}
  321 & -160 \\
  480 & -239 \\
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
  -3 \cdot (-3)^k + 4 \cdot (-2)^k & 6 \cdot (-3)^k - 6 \cdot (-2)^k \\
  -2 \cdot (-3)^k + 2 \cdot (-2)^k & 4 \cdot (-3)^k - 3 \cdot (-2)^k \\
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
  3 & 0 & -1 \\
  0 & 1 & 0 \\
\end{bmatrix}; \quad \lambda = 4: \begin{bmatrix} 1 \\
  0 \\
\end{bmatrix}
\]

When an answer involves a diagonalization, \(A = PDP^{-1}\), the factors \(P\) and \(D\) are not unique, so your answer may differ from that given here.

8. Not diagonalizable. The eigenvalue 3 has multiplicity two, but the associated eigenspace is only one-dimensional.

10. \[
P = \begin{bmatrix}
  -1 & 3 \\
  1 & 4 \\
\end{bmatrix}, \quad D = \begin{bmatrix}
  -2 & 0 \\
  0 & 5 \\
\end{bmatrix}
\]

12. \[
P = \begin{bmatrix}
  -1 & -1 & 1 \\
  1 & 0 & 1 \\
  0 & 1 & 1 \\
\end{bmatrix}, \quad D = \begin{bmatrix}
  2 & 0 & 0 \\
  0 & 2 & 0 \\
  0 & 0 & 5 \\
\end{bmatrix}
\]