has four distinct eigenvalues, all close to zero. However, the only eigenvalue is 0, with multiplicity 4, because $A^4 = 0$.

Chapter 6

Section 6.1, page 336

2. 35. 5, 4

4. $\begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$

6. $\begin{bmatrix} 30/49 \\ -10/49 \end{bmatrix}$

8. 7

10. $\begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$

12. $\begin{bmatrix} .8 \\ .6 \end{bmatrix}$

14. $2\sqrt{17}$

16. Orthogonal

18. Not orthogonal

19. a. True. See the definition of $\|v\|$.  
   b. True. See Theorem 1(c).  
   c. True. See the discussion of Fig. 5.  
   d. False. Counterexample: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.  
   e. True. See the box following Example 6.

20. a. True. See Example 1 and Theorem 1(a).  
   b. False. The absolute value is missing. See the box before Example 2.  
   c. True, by definition of the orthogonal complement.  
   d. True, by the Pythagorean Theorem.  
   e. True, by Theorem 3.

22. $u \cdot u \geq 0$ because $u \cdot u$ is a sum of squares of the entries in $u$.  
The sum of squares of numbers is zero if and only if all the numbers are themselves zero.

24. $\|u + v\|^2 = (u + v) \cdot (u + v) = u \cdot u + 2u \cdot v + v \cdot v$  
   $\|u - v\|^2 = (u - v) \cdot (u - v) = u \cdot u - 2u \cdot v + v \cdot v$  
   $\|u\|^2 = 2u \cdot v + v \cdot v$  
   When $\|u + v\|^2$ and $\|u - v\|^2$ are added, the $u \cdot v$ terms cancel, and the result is $2\|u\|^2 + 2\|v\|^2$.

26. Theorem 2 in Chapter 4, because $W$ is the null space of the $1 \times n$ matrix $u^T$.  
    $W$ is a plane through the origin of $\mathbb{R}^3$.

28. An arbitrary $w$ in $\text{Span} \{u, v\}$ has the form $w = c_1 u + c_2 v$.  
    If $y$ is orthogonal to $u$ and $v$, then $u \cdot y = 0$ and $v \cdot y = 0$.  
    By linearity of the inner product [Theorem 1(b) and 1(c)],  
    $w \cdot y = (c_1 u + c_2 v) \cdot y = c_1 u \cdot y + c_2 v \cdot y = c_1 0 + c_2 0 = 0$.

30. a. If $z$ is in $W^\perp$, $u$ is in $W$, and $c$ is any scalar, then  
    $(cz) \cdot u = c(z \cdot u) = c 0 = 0$, since $u$ is any element of $W$, $cz$ is in $W^\perp$.
   b. Take any $z_1, z_2$ in $W^\perp$. Then, for any $u$ in $W$,  
    $(z_1 + z_2) \cdot u = z_1 \cdot u + z_2 \cdot u = 0 + 0 = 0$, which shows that $z_1 + z_2$ is in $W^\perp$.

32. [M] This exercise anticipates Theorem 7 in Section 6.2.  
The matrix $A$ has orthonormal columns.

33. [M] The mapping $x \mapsto T(x) = (\begin{bmatrix} x \cdot x \\ x \cdot v \end{bmatrix})$ is a linear transformation.  
In Section 6.2, the mapping will be called the orthogonal projection of $x$ onto $\text{Span} \{v\}$.  
To verify the linearity, take any $x$ and $y$ in $\mathbb{R}^4$ (or $\mathbb{R}^n$) and any scalar $c$.  
Then properties of the inner product (Theorem 1) show that  
$T(x + y) = (x + y) \cdot v = x \cdot v + y \cdot v$  
$= (x \cdot v + y \cdot v) = T(x) + T(y)$  
$T(cx) = c(x \cdot v) = c(x \cdot v) = cT(x)$

Another argument is to view $T$ as the composition of three linear mappings:  
$x \mapsto a = x \cdot v, a \mapsto b = a/(v \cdot v),$  
and $b \mapsto bv$.

34. [M] $N = \begin{bmatrix} -5 & 1 \\ -1 & 4 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$.

Section 6.2, page 344

2. Orthogonal  
4. Orthogonal  
6. Not orthogonal

8. Show $u_1 \cdot u_2 = 0$, mention Theorem 4, and observe that two linearly independent vectors in $\mathbb{R}^2$ form a basis. Then obtain  
$x = \begin{bmatrix} 15 \\ 10 \end{bmatrix} + \begin{bmatrix} 30 \\ 40 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

10. Show $u_1 \cdot u_2 = 0, u_1 \cdot u_3 = 0$, and $u_1 \cdot u_3 = 0$. Mention Theorem 4, and observe that three linearly independent vectors in $\mathbb{R}^3$ form a basis. Then obtain  
$x = \frac{3}{10} u_1 + \frac{3}{10} u_2 + \frac{3}{10} u_3 = \frac{3}{2} u_1 + \frac{1}{2} u_2 + \frac{1}{2} u_3$

12. $\begin{bmatrix} .4 \\ -1.2 \end{bmatrix}$  
14. $y = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$

16. $y - \hat{y} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$, distance is $\sqrt{45} = 3\sqrt{5}$

18. Not orthogonal

22. Orthogonal  
24. True  
26. True by definition of $\|v\|$.  
28. True by Theorem 3

29. If $v_1, \ldots, v_n$ are linearly independent, then $\text{Span} \{v_1, \ldots, v_n\}$ is a subspace of $\mathbb{R}^n$.

30. If $U$ is an orthonormal basis for $\mathbb{R}^n$, and $\text{Span} \{u_1, \ldots, u_n\}$ is also an orthonormal basis, then $U$ and $\text{Span} \{u_1, \ldots, u_n\}$ are the same set.

32. [M] This exercise anticipates Theorem 7 in Section 6.2.  
The matrix $A$ has orthonormal columns.
Section 6.3

18. Not orthogonal

\[ \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \quad \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} \]

20. (0.1)

22. Orthonormal

23. a. True. For example, the vectors \( \mathbf{u} \) and \( \mathbf{y} \) in Example 3 are linearly independent but not orthogonal.

b. True. The formulas for the weights are given in Theorem 5.

c. False. See the paragraph following Example 5.

d. False. The matrix must also be square. See the paragraph before Example 7.

e. False. See Example 4. The distance is \( \| \mathbf{y} - \mathbf{y} \| \).


b. False. To be orthonormal, the vectors in \( S \) must be unit vectors as well as being orthogonal to each other.

c. True. See Theorem 7(a).

d. True. See the paragraph before Example 3.

e. True. See the paragraph before Example 7.

26. If \( v_1, \ldots, v_n \) are nonzero and orthogonal, then they are linearly independent, by Theorem 4. By the Invertible Matrix Theorem, \( \{v_1, \ldots, v_n\} \) is a basis for \( \mathbb{R}^n \). If \( W = \text{Span} \{v_1, \ldots, v_n\} \), then \( W \) must be \( \mathbb{R}^n \).

28. If \( U \) is an \( n \times n \) orthogonal matrix, then \( T = U U^{-1} = U^T \). Since \( U \) is the transpose of \( U^T \), Theorem 6 applied to \( U^T \) says that \( U^T \) has orthonormal columns. In particular, the columns of \( U^T \) are linearly independent and hence form a basis for \( \mathbb{R}^n \), by the Invertible Matrix Theorem (see Theorem 4.6). That is, the rows of \( U \) form a basis (in fact, an orthonormal basis) for \( \mathbb{R}^n \).

30. If \( U \) is an orthogonal matrix, its columns are orthonormal. Interchanging the columns does not change their orthonormality, so the new matrix—say, \( V \)— still has orthonormal columns. By Theorem 6, \( V^TV = I \). Since \( V \) is square, \( V^T = V^{-1} \) by the Invertible Matrix Theorem.

32. If \( v_1 \cdot v_2 = 0 \), then by Theorem 1(c) in Section 6.1, \( (c_1 v_1) \cdot (c_2 v_2) = c_1 c_2 (v_1 \cdot v_2) = 0 \).

34. Let \( L = \text{Span} \{ \mathbf{u} \} \), where \( \mathbf{u} \) is nonzero, and let \( T(\mathbf{y}) = \text{refl}_L \mathbf{y} = 2 \cdot \text{proj}_L \mathbf{y} - \mathbf{y} \). By Exercise 33, the mapping \( \mathbf{y} \mapsto \text{proj}_L \mathbf{y} \) is linear. Thus, for \( \mathbf{y} \) and \( \mathbf{z} \) in \( \mathbb{R}^n \) and any scalars \( c \) and \( d \),

\[
T(c \mathbf{y} + d \mathbf{z}) = 2 \cdot \text{proj}_L(c \mathbf{y} + d \mathbf{z}) - (c \mathbf{y} + d \mathbf{z})
= 2c \cdot \text{proj}_L \mathbf{y} + 2d \cdot \text{proj}_L \mathbf{z} - c \mathbf{y} - d \mathbf{z}
= cT(\mathbf{y}) + dT(\mathbf{z})
\]

Thus \( T \) is linear.

35. [M] The proof of Theorem 6 shows that the inner products to be checked are actually entries in the matrix product \( A^T A \). A calculation shows that \( A^T A = 100 I_4 \). Since the off-diagonal entries in \( A^T A \) are zero, the columns of \( A \) are orthogonal.

36. [M] a. \( U^T U = I_n \), but \( U U^T \) is an \( 8 \times 8 \) matrix which is nothing like \( I_8 \). In fact

\[
U U^T = \begin{bmatrix}
82 & 0 & -20 & 8 & 6 & 20 & 24 & 0 \\
0 & 42 & 24 & 0 & -20 & 6 & 20 & -32 \\
-20 & 24 & 58 & 20 & 0 & 32 & 0 & 6 \\
8 & 0 & 20 & 82 & 24 & -20 & 6 & 0 \\
6 & -20 & 0 & 24 & 18 & 0 & -8 & 20 \\
20 & 6 & 32 & -20 & 0 & 58 & 0 & 24 \\
24 & 20 & 0 & 6 & -8 & 0 & 18 & -20 \\
0 & -32 & 6 & 0 & 20 & 24 & -20 & 42 \\
\end{bmatrix}
\]

b. The vector \( \mathbf{p} = U U^T \mathbf{y} \) is in \( \text{Col} \mathbf{U} \) because \( \mathbf{p} = U(U^T \mathbf{y}) \). Since the columns of \( U \) are simply scaled versions of the columns of \( A \), \( \text{Col} \mathbf{U} = \text{Col} \mathbf{A} \). Thus \( \mathbf{p} \) is in \( \text{Col} \mathbf{A} \).

d. From part (c), \( \mathbf{z} \) is orthogonal to each column of \( A \). By Exercise 29 in Section 6.1, \( \mathbf{z} \) must be orthogonal to every vector in \( \text{Col} \mathbf{A} \); that is, \( \mathbf{z} \) is in \( \text{(Col} \mathbf{A})^\perp \).

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Section 6.3, page 352

2. \( \mathbf{v} = 2 \mathbf{u}_1 + \frac{3}{2} \mathbf{u}_2 + \frac{2}{3} \mathbf{u}_3 - \frac{5}{8} \mathbf{u}_4 \); \( \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ -5 \end{bmatrix} \)

4. \( \begin{bmatrix} 3 \\ 6 \end{bmatrix} \) \( \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \mathbf{y} \) \( \begin{bmatrix} 3/2 \\ 7/2 \end{bmatrix} + \begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix} \)

10. \( \mathbf{y} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \) \( \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix} \)

14. \( \begin{bmatrix} 1 \\ 0 \\ -1/2 \\ -3/2 \end{bmatrix} \) \( \begin{bmatrix} 16 \\ 8 \end{bmatrix} \)

18. a. \( U^T U = [I] = I, \) \( U U^T = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} \)

b. \( \text{proj}_I \mathbf{y} = \frac{20}{\sqrt{10}} \mathbf{u}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \)

\( (U U^T) \mathbf{y} = \begin{bmatrix} -2.7 & -2.7 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \)

20. Any multiple of \( \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix} \), such as \( \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \)

21. a. True. See the calculations for \( z_3 \) in Example 1 or the box after Example 6 in Section 6.1.

b. True, by the Orthogonal Decomposition Theorem.