SECTION 5.2 The Pigeonhole Principle

2. This follows from the pigeonhole principle, with $d = 20$.

4. We assume that the woman does not replace the balls after drawing them.

a) There are two colors: there are the pigeons. We want to know how many pigeons would make at least one color be at least 6. If five balls are selected, at least $\lceil 5/2 \rceil = 3$ must have the same color. On the other hand, if the woman draws 12 fewer balls, then 10 of them could be red, and she might not get her three blue balls. This time the number of balls did matter.

b) There must be only 3 possible combinations when the 15 is divided by 4, namely 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1.

the pigeonhole principle, if we have 4 + 1 residuals, then at least two must be the same.

8. This is just a re-statement of the pigeonhole principle, with $b = [7].$

10. The midpoint of the sequence whose midpoints are $(2, 1), (3, 2), (4, 3), (5, 4), (6, 5), (7, 6),$ and $(8, 7)$ are $(4, 5)$ and $(5, 6)$ (both even), and $(2, 3)$ and $(3, 4)$ (both odd). What matters in this problem is the position of the coordinates. There are four possible pairs of points $(a, b), (b, a)$ or $(a, a), (a, a)$. Since we are given five points, the pigeonhole principle guarantees that at least two of them will have the same pair of points. The midpoint of the sequence joining these two points will therefore have integer coordinates.

12. This is similar in spirit to Exercise 10. Working modulo 5, there are 25 pairs: $(0, 0), (0, 1), \ldots, (4, 4).$ There will be 25 ordered pairs of integers (a, b) such that no two of them were equal when reduced modulo 5. The pigeonhole principle guarantees that if we have 26 such points, then at least two of them will have the same coordinates, modulo 5.

14. a) We can group the first two positive integers into five subsequences of two integers each, such that adding two integers gives a multiple of 10. We could have the pigeonhole principle at least two of them were equal. Moreover, if we forget about those two in the same group, then there are five more integers and four groups again the pigeonhole principle. In each case, these two integers have a sum of 11, as desired.

b) No. The set $(1, 2, 3, 4, 5)$ has only 2 and from the same group, so the only pair with sum 11 is 5 and 6.
36. Let $K(x)$ be the number of other people at the party that person $x$ knows. The possible values for $K(x)$ are $0, 1, \ldots, n - 1$, where $n \geq 2$ is the number of people at the party. We cannot apply the pigeonhole principle directly, since there are $n$ pigeons and $n$ pigeonholes. However, it is impossible for both 0 and $n - 1$ to be in the range of $K$, since if one person knows everybody else, then nobody can know no one else (we assume that “knowing” is symmetric). Therefore the range of $K$ has at most $n - 1$ elements, whereas the domain has $n$ elements, so $K$ is not one-to-one, precisely what we wanted to prove.

38. a) The solution of Exercise 37, with 24 replaced by 2 and 149 replaced by 127, tells us that the statement is true.

b) The solution of Exercise 37, with 24 replaced by 33 and 149 replaced by 148, tells us that the statement is true.

c) We begin in a manner similar to the solution of Exercise 37. Look at $a_1$, $a_2$, $\ldots$, $a_{75}$, $a_1 + 25$, $\ldots$, $a_{75} + 25$, where $a_i$ is the total number of matches played up through hour $i$. Then $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$, and $25 \leq a_1 + 25 < a_2 + 25 < \cdots < a_{75} + 25 \leq 150$. Now either these 150 numbers are precisely all the number from 1 to 150, or else by the pigeonhole principle we get, as in Exercise 37, $a_i = a_j + 25$ for some $i$ and $j$ and we are done. In the former case, however, since each of the numbers $a_i + 25$ is greater than or equal to 25, the numbers $1, 2, \ldots, 25$ must all appear among the $a_i$’s. But since the $a_i$’s are increasing, the only way this can happen is if $a_1 = 1, a_2 = 2, \ldots, a_{25} = 25$. Thus there were exactly 25 matches in the first 25 hours.

d) We need a different approach for this part, an approach, incidentally, that works for many numbers besides 30 in this setting. Let $a_1$, $a_2$, $\ldots$, $a_{75}$ be as before, and note that $1 \leq a_1 < a_2 < \cdots < a_{75} \leq 125$. By the pigeonhole principle two of the numbers among $a_1$, $a_2$, $\ldots$, $a_{75}$ are congruent modulo 30. If they differ by 30, then we have our solution. Otherwise they differ by 50 or more, so $a_{75} - a_1 \geq 61$. Similarly, among $a_{75}$ through $a_{13}$, either we find a solution, or two numbers must differ by 60 or more; therefore we can assume that $a_{75} \geq 121$. But this means that $a_{75} \geq 126$, a contradiction.

46. Look at the pigeonholes $\{1000, 1001\}$, $\{1002, 1003\}$, $\{1004, 1005\}$, $\ldots$, $\{1038, 1099\}$. There are clearly 50 sets in this list. By the pigeonhole principle, if we have 51 numbers in the range from 1000 to 1099 inclusive, then at least two of them must come from the same set. These are the desired two consecutive house numbers.

42. Suppose this statement were not true. Then for each $i$, the $i^{th}$ box contains at most $n_i - 1$ objects. Adding we have at most $(n_1 - 1) + (n_2 - 1) + \cdots + (n_t - 1) = n_1 + n_2 + \cdots + n_t - t$ objects in all, contradicting the fact that there were $n_1 + n_2 + \cdots + n_t - t + 1$ objects in all. Therefore the statement must be true.