SETS

Sections 2.1, 2.2 and 2.4
Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities
Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.
Sets

• A set is an unordered collection of objects.
  • the students in this class
  • the chairs in this room
• The objects in a set are called the elements, or members of the set. A set is said to contain its elements.
• The notation $a \in A$ denotes that $a$ is an element of the set $A$.
• If $a$ is not a member of $A$, write $a \notin A$
Describing a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important
  
  \[ S = \{a,b,c,d\} = \{b,c,a,d\} \]

- Each distinct object is either a member or not; listing more than once does not change the set.
  
  \[ S = \{a,b,c,d\} = \{a,b,c,b,c,d\} \]

- Elipses (...) may be used to describe a set without listing all of the members when the pattern is clear.
  
  \[ S = \{a,b,c,d, \ldots, z\} \]
Roster Method

- Set of all vowels in the English alphabet:
  \[ V = \{a,e,i,o,u\} \]
- Set of all odd positive integers less than 10:
  \[ O = \{1,3,5,7,9\} \]
- Set of all positive integers less than 100:
  \[ S = \{1,2,3,\ldots,99\} \]
- Set of all integers less than 0:
  \[ S = \{\ldots,-3,-2,-1\} \]
Some Important Sets

\[ N = \text{natural numbers} = \{0,1,2,3,\ldots\} \]
\[ Z = \text{integers} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\} \]
\[ Z^+ = \text{positive integers} = \{1,2,3,\ldots\} \]
\[ R = \text{set of real numbers} \]
\[ R^+ = \text{set of positive real numbers} \]
\[ C = \text{set of complex numbers} \]
\[ Q = \text{set of rational numbers} \]
Set-Builder Notation

• Specify the property or properties that all members must satisfy:
  
  \[ S = \{ x \mid x \text{ is a positive integer less than 100} \} \]
  
  \[ O = \{ x \mid x \text{ is an odd positive integer less than 10} \} \]
  
  \[ O = \{ x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10 \} \]

• A predicate may be used:
  
  \[ S = \{ x \mid P(x) \} \]

• Example: \[ S = \{ x \mid \text{Prime}(x) \} \]

• Positive rational numbers:
  
  \[ Q^+ = \{ x \in \mathbb{R} \mid x = p/q, \text{ for some positive integers } p, q \} \]
Interval Notation

\[ [a,b] = \{ x \mid a \leq x \leq b \} \]
\[ [a,b) = \{ x \mid a \leq x < b \} \]
\[ (a,b] = \{ x \mid a < x \leq b \} \]
\[ (a,b) = \{ x \mid a < x < b \} \]

*closed interval*  \([a,b]\)

*open interval*  \((a,b)\)
Universal Set and Empty Set

- The *universal set* $U$ is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized $\emptyset$, but \{} is also used.
Subsets

**Definition:** The set $A$ is a *subset* of $B$, if and only if every element of $A$ is also an element of $B$.

- The notation $A \subseteq B$ is used to indicate that $A$ is a subset of the set $B$.
- $A \subseteq B$ holds if and only if $\forall x (x \in A \rightarrow x \in B)$ is true.

Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set $S$.
Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set $S$. 
Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that \( A \subseteq B \), show that if \( x \) belongs to \( A \), then \( x \) also belongs to \( B \).

- **Showing that A is not a Subset of B**: To show that \( A \) is not a subset of \( B \), \( A \nsubseteq B \), find an element \( x \in A \) with \( x \notin B \). (Such an \( x \) is a counterexample to the claim that \( x \in A \) implies \( x \in B \).)

**Examples:**

1. The set of all computer science majors at your school is a subset of all students at your school.
Another look at Equality of Sets

• Recall that two sets $A$ and $B$ are equal, denoted by $A = B$, iff
  $\forall x (x \in A \leftrightarrow x \in B)$

• Using logical equivalences we have that $A = B$ iff
  $\forall x [(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)]$

• This is equivalent to
  $A \subseteq B$ and $B \subseteq A$
Proper Subsets

**Definition:** If $A \subseteq B$, but $A \neq B$, then we say $A$ is a *proper subset* of $B$, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x (x \in A \rightarrow x \in B) \land \exists x (x \in B \land x \notin A)$$

is true.

Venn Diagram
Set Cardinality

**Definition:** If there are exactly \( n \) distinct elements in \( S \) where \( n \) is a nonnegative integer, we say that \( S \) is *finite*. Otherwise it is *infinite*.

**Definition:** The *cardinality* of a finite set \( A \), denoted by \(|A|\), is the number of (distinct) elements of \( A \).

**Examples:**

\(|\emptyset| = 0\)

Let \( S \) be the letters of the English alphabet. Then \(|S| = 26\)

\(|\{1,2,3\}| = 3\)

The set of integers is infinite.
Power Sets

**Definition:** The set of all subsets of a set $A$, denoted $P(A)$, is called the *power set* of $A$.

**Example:** If $A = \{a,b\}$ then

$$P(A) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$

- If a set has $n$ elements, then the cardinality of the power set is $2^n$.
  - If $B = \{1, 2, 3, 4\}$
  - $|B|=4$
  - $|P(B)| = 2^4=16$
  - $\{1,2,4\} \subseteq B$ and $\{1,2,4\} \in P(B)$
  - $\{4\} \subseteq B$ and $\{4\} \in P(B)$
  - $4 \in B$ but $4 \notin P(B)$
Union

• **Definition**: Let \( A \) and \( B \) be sets. The *union* of the sets \( A \) and \( B \), denoted by \( A \cup B \), is the set:

\[
\{ x \mid x \in A \lor x \in B \}
\]

• **Example**: What is \( \{1,2,3\} \cup \{3,4,5\} \)?

**Solution**: \( \{1,2,3,4,5\} \)

![Venn Diagram for \( A \cup B \)]
Intersection

- **Definition**: The *intersection* of sets $A$ and $B$, denoted by $A \cap B$, is
  \[
  \{ x \mid x \in A \land x \in B \}
  \]

- Note if the intersection is empty, then $A$ and $B$ are said to be *disjoint*.

- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?
  Solution: $\{3\}$

- **Example**: What is?
  $\{1,2,3\} \cap \{4,5,6\}$?
  Solution: $\emptyset$

- Venn Diagram for $A \cap B$
**Complement**

**Definition:** If $A$ is a set, then the complement of the $A$ (with respect to $U$), denoted by $\bar{A}$ is the set $U - A$

$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of $A$ is sometimes denoted by $A^c$.)

**Example:** If $U$ is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$
**Difference**

- **Definition**: Let $A$ and $B$ be sets. The *difference* of $A$ and $B$, denoted by $A - B$, is the set containing the elements of $A$ that are not in $B$. The difference of $A$ and $B$ is also called the complement of $B$ with respect to $A$.

\[
A - B = \{ x \mid x \in A \land x \notin B \} = A \cap \overline{B}
\]

![Venn Diagram for $A - B$]
Combining set operations

\[(A \cap B) \cup C\]
Combining set operations

\[(A \cap B) \cup C\]
Combining set operations

\((A \cap B) \cup C\)
Combining set operations

\[(A \cap B) \cup C\]
Example: \( U = \{0,1,2,3,4,5,6,7,8,9,10\} \quad A = \{1,2,3,4,5\}, \quad B = \{4,5,6,7,8\} \)

1. \( A \cup B \)
   \[ \text{Solution: } \{1,2,3,4,5,6,7,8\} \]

2. \( A \cap B \)
   \[ \text{Solution: } \{4,5\} \]

3. \( \bar{A} \)
   \[ \text{Solution: } \{0,6,7,8,9,10\} \]

4. \( \bar{B} \)
   \[ \text{Solution: } \{0,1,2,3,9,10\} \]

5. \( A - B \)
   \[ \text{Solution: } \{1,2,3\} \]

6. \( B - A \)
   \[ \text{Solution: } \{6,7,8\} \]
Set Identities

- Identity laws \( A \cup \emptyset = A \quad A \cap U = A \)
- Domination laws \( A \cup U = U \quad A \cap \emptyset = \emptyset \)
- Idempotent laws \( A \cup A = A \quad A \cap A = A \)
- Complementation law \( \overline{(A)} = A \)

Continued on next slide →
Set Identities

- Commutative laws
  \[ A \cup B = B \cup A \quad A \cap B = B \cap A \]

- Associative laws
  \[ A \cup (B \cup C) = (A \cup B) \cup C \]
  \[ A \cap (B \cap C) = (A \cap B) \cap C \]

- Distributive laws
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

Continued on next slide
Set Identities

• De Morgan’s laws
  \[
  \overline{A \cup B} = \overline{A} \cap \overline{B} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}
  \]

• Absorption laws
  \[
  A \cup (A \cap B) = A \quad A \cap (A \cup B) = A
  \]

• Complement laws
  \[
  A \cup \overline{A} = U \quad A \cap \overline{A} = \emptyset
  \]
Set Partitions

A partition of set $A$ is a set of sets $B_1, \ldots, B_n$

- Each $B_i$ is non-empty for $i \in \{1, \ldots, n\}$
- Each pair is disjoint: $B_i \cap B_j \neq \emptyset$
  - for $i, j \in \{1, \ldots, n\}$
- Their union is $A$:
  - $B_1 \cup \ldots \cup B_n = A$

- Example: partition of $\mathbb{R}$
  - $(\infty, -2]$
  - $(-2, 3]$
  - $(3, \infty)$