Homework 2
Sections 1.4-1.6

Please write your name and student ID number clearly at the top of your homework.
If you have multiple pages, please make sure they are secured together.

You should turn in your homework to the drop box located on the 3rd floor of Bren Hall, around
the corner from room 3013.

Problem 1
Use a truth table to prove the distributive law: \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \)

Problem 2
Show that the proposition \((p \land \neg q) \rightarrow r\) is neither a tautology or a contradiction.

Problem 3
Use a truth table to show that \((p \leftrightarrow q) \lor (p \leftrightarrow \neg q)\) is a tautology.

Problem 4
Use the laws of logical equivalence to prove the following:
   a. \( \neg p \rightarrow \neg q \equiv q \rightarrow p \)
   b. \( p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q) \)
   c. \( \neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \lor r) \)
   d. \( (p \rightarrow r) \lor (q \rightarrow r) \equiv (p \land q) \rightarrow r \)

Problem 5
In the following question, the domain is a set of male patients in a clinical study. Define the
following predicates to be:

\[ P(x): \text{x was given the placebo} \]
\[ D(x): \text{x was given the medication} \]
\[ A(x): \text{x had fainting spells} \]
\[ M(x): \text{x had migraines} \]

Express the following statements in logic. If your quantified statement has a negation in front of
the quantifier, use De Morgan's law to move the negation after the quantifier. (You can assume
that there is exactly one patient in the study named Sam.)

   a. None of the patients had migraines
   b. Every patient was given the placebo or the medication.
   c. No patient was given both the medication and a placebo.
   d. There was a patient who had fainting spells even though he was given the placebo
There was a patient who had migraines and a patient who had fainting spells.  
Every patient who had fainting spells and migraines was given the medication.  
Every patient who was given the medication had fainting spells or migraines.  
Sam was given the medication and did not have migraines or fainting spells.

Problem 6  
This problem refers to the same predicates as in the previous problem.

Suppose that there are five patients who participated in the study. The table below shows the names of the patients and the truth value for each patient and each predicate:

<table>
<thead>
<tr>
<th></th>
<th>P(x)</th>
<th>D(x)</th>
<th>A(x)</th>
<th>M(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frodo</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>Gandalf</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>Gimli</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>Aragorn</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>Bilbo</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

For each of the following quantified statements, indicate whether the statement is a proposition. If the statement is a proposition, give its truth value and translate the expression into English.

a. $\exists x (M(x) \land D(x))$

b. $\exists x M(x) \land \exists x D(x)$

c. $\exists x M(x) \land D(x)$

d. $\forall x (A(x) \lor M(x))$

e. $\neg \exists x M(x)$

f. $\neg M(Frodo)$

g. $\forall x (D(x) \leftrightarrow M(x))$

h. $\forall x ((M(x) \land A(x)) \rightarrow D(x))$

i. $\exists x ((M(x) \land A(x)) \rightarrow D(x))$
Problem 7
Define the domain of discourse for two variables \( x \) and \( y \) to be the set of musicians in an orchestra. Define the following predicates:

- \( S(x) \): \( x \) plays a stringed instrument.
- \( B(x) \): \( x \) plays a brass instrument.
- \( P(x, y) \): \( x \) practices harder than \( y \)

Give a quantified expression that is equivalent to the following English statements:

a. There are no brass players in the orchestra.
b. All the string players practice harder than all the brass players.
c. Sam practices more than anyone else in the orchestra.
d. Exactly one person practices more than Sam.
e. There is someone in the orchestra who plays a stringed instrument and a brass instrument.
f. Everyone practices more than Nancy.
g. There is a brass player who practices harder than all the string players.

Problem 8
Write the negation of each of the following logical expressions so that all negations appear after the quantifiers.

a. \( \forall x \exists y \exists z P(y, x, z) \)
b. \( \exists x \exists y P(x, y) \land \forall x \forall y Q(x, y) \)
c. \( \exists x \forall y (P(x, y) \leftrightarrow P(y, x)) \)
d. \( \exists x \forall y (P(x, y) \rightarrow Q(x, y)) \)

Problem 9
Determine the truth value of each expression below if the domain is the set of all real numbers.

a. \( \forall x \exists y (xy > 0) \)
b. \( \exists x \forall y (xy = 0) \)
c. \( \forall x \exists y y^2 = x \)
d. \( \forall x \forall y \exists z (z = (x - y)/3) \)
e. \( \forall x \forall y (xy = yx) \)
f. \( \exists x \exists y \exists z (x^2 + y^2 = z^2) \)

Problem 10
The domain of discourse is the members of a chess club. The predicate \( B(x, y) \) means that person \( x \) has beaten person \( y \) at some point in time. Give a logical expression equivalent to the following English statements.

a. Everyone has been beaten before.
b. No one has ever beat Nancy.
c. Everyone has won at least one game.
d. No one has beaten both Ingrid and Dominic.
e. There are two members who have never been beaten.

Problem 11
The domain of discourse for this problem is a group of three people who are working on a project. To make notation easier, the people are numbered 1, 2, 3. The predicate $M(x, y)$ indicates whether $x$ has sent an email to $y$, so $M(2, 3)$ is read "Person 2 has sent an email to person 3." The table below shows the value of the predicate $M(x,y)$ for each $(x,y)$ pair. The truth value in row $x$ and column $y$ gives the truth value for $M(x,y)$.

\[
\begin{array}{|c|c|c|}
\hline
M & 1 & 2 & 3 \\
\hline
1 & T & T & T \\
2 & T & F & T \\
3 & T & T & F \\
\hline
\end{array}
\]

For each of the quantified statements, indicate whether the statement is true.

a. $\forall x \forall y M(x,y)$
b. $\forall x \forall y ((x \neq y) \rightarrow M(x,y))$
c. $\exists x \exists y \neg M(x,y)$
d. $\exists x \exists y ((x \neq y) \land \neg M(x,y))$
e. $\forall x \exists y \neg M(x,y)$
f. $\exists x \forall y M(x,y)$