Image/signal Processing

Note Title 9/5/2008

What are signals?

Functions - can be simple or multi-dimensional.

$y = f(t)$ → one-dimensional signal.
e.g. audio

$I = f(x, y)$ → 2 dimensional signal
e.g. image

$M = f(x, y, z)$ → 3 dimensional signal.
E.g. 3D Model
M defines color at 3D location (x, y, z)

Or, Video, where z = time.

Systems

Any black box that modifies a signal. Eg. audio amplifier.

image processing algorithm, and so on.

Systems can be very complex.

Usually we will deal with a class of simpler systems, called <u>Linear Systems</u>.

Linear Systems have some good properties.

a) Homogeneity

If $x(t) \rightarrow \boxed{S} \rightarrow y(t)$

Then $kx(t) \rightarrow \boxed{S} \rightarrow ky(t)$

b) Additivity

If $x_1(t) \rightarrow \boxed{S} \rightarrow y_1(t)$

$x_2(t) \rightarrow \boxed{S} \rightarrow y_2(t)$

then $x_1(t) + x_2(t) \rightarrow \boxed{S} \rightarrow y_1(t) + y_2(t)$

$\therefore$ Each signal passed independently, no interaction between them.

c) Shift Invariance

If $x(t) \rightarrow \boxed{S} \rightarrow y(t)$

then $x(t+s) \rightarrow \boxed{S} \rightarrow y(t+s)$

Some other properties follow from the above.

① If $x(t) \rightarrow \boxed{\boxed{S_A} \rightarrow \boxed{S_B}} \rightarrow y(t)$

Cascaded Sys.

then $x(t) \rightarrow \boxed{\boxed{S_B} \rightarrow \boxed{S_A}} \rightarrow y(t)$

COMMUTATIVE → order of application does not matter

② SUPERPOSITION (Multiple i/o)

If $x_1(t) \rightarrow \boxed{S} \rightarrow y_1^1(t)$, $\rightarrow y_2^1(t)$

and $x_2(t) \rightarrow \boxed{S} \rightarrow y_1^2(t)$, $\rightarrow y_2^2(t)$

then $x_1(t) + x_2(t) \rightarrow \boxed{S} \rightarrow y_1^1(t) + y_1^2(t)$, $\rightarrow y_2^1(t) + y_2^2(t)$

How is this property important?

It can help in finding the response of a system to complex signals. (can scale to n)

<u>Three steps</u>:

$$
\begin{array}{c}
x(t) \\
\Big\Downarrow \text{ DECOMPOSITION} \\
x_1(t),\ x_2(t)\ \cdots\ x_n(t) \quad \text{s.t } \sum x_i(t) = x(t) \\
\downarrow \qquad \downarrow \qquad\qquad \downarrow \\
y_1(t) \quad y_2(t)\ \cdots\ y_n(t) \\
\Big\Downarrow \text{ SYNTHESIS} \\
y(t) \quad \text{s.t } \sum y_i(t) = y(t)
\end{array}
$$

Each $x_i(t)$ is a much simpler signal to which it is easy to find the response of S, say $y_i(t)$.

$\therefore$ What are the different ways to decompose?

Many ways $\longrightarrow$ we will study two in this class.

<u>Discrete Signals</u>

Analog Signal

<u>Sample & hold</u> to measure signal at some discrete values.

Eg.

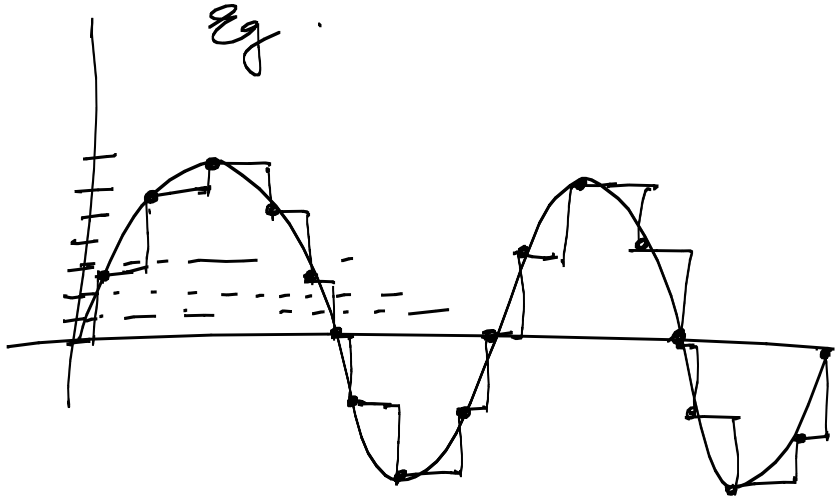
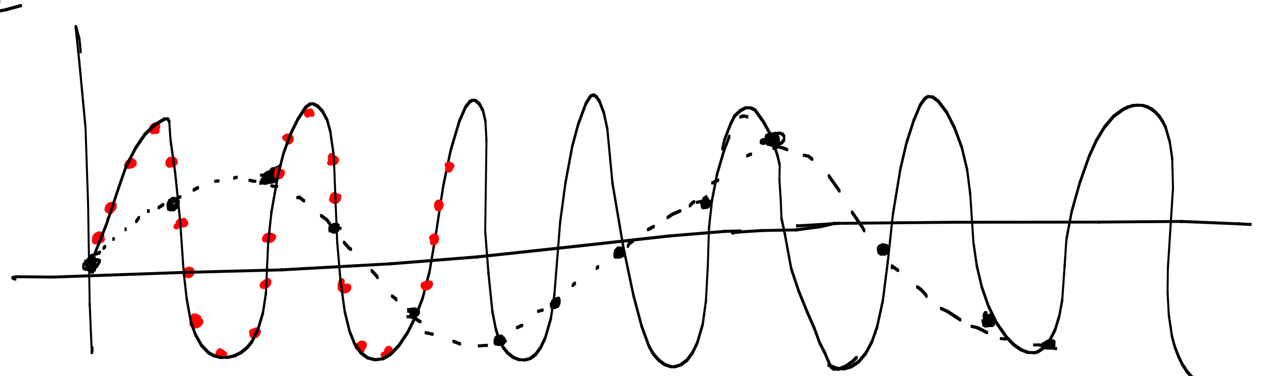


Fig 1

∴ This is a discrete version of analog signal. ∴ Now the question is, how often should we sample so that we can reconstruct the signal.

Nyquist's Sampling theorem -
At least twice the maximum frequency of the signal.

Eg.



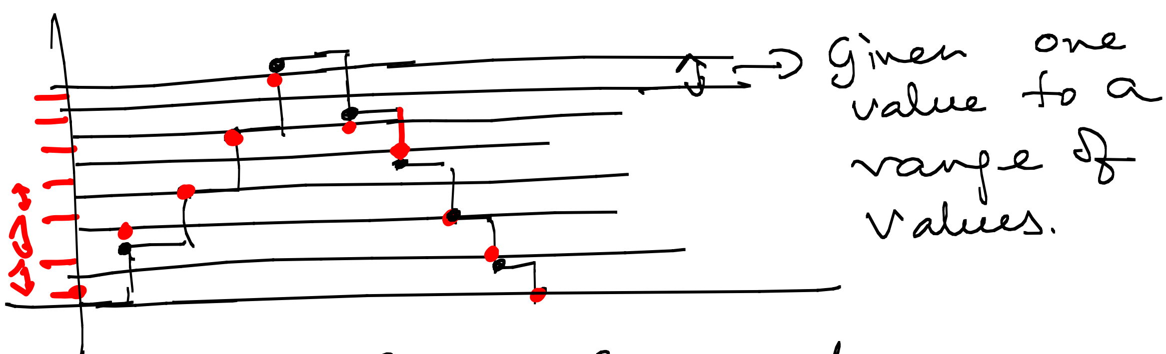
Inadequate sampling, hence you get an imposter freq. Called aliasing.

But if you sampled adequately, you would have got the correct wave.

QUANTIZATION

Sampling is due to the fact that we cannot sample all the values of the independent axis.

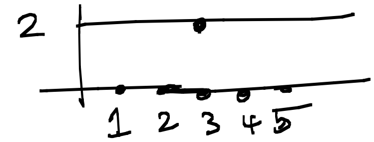
Same is true for the dependent axis. We do not have infinite precision. ∴ limited # of bits.



∴ Each sample has an error and is not represented accurately. Called Quantization error. If the step sizes are uniform, it is ½ step size. If non-uniform, maximum error is ½ the maximum step size.

CONVOLUTION

Impulse :- A discrete signal with only one non-zero sample.



$i[t]$

Delta - Is an impulse whose non-zero sample is at 0 & has value 1.

$\delta[t]$

$\therefore i[t] = 2\,\delta[t-3]$

Any impulse can be represented as a shifted scaled delta. General form $i[t] = k\,\delta[t-s]$

Simplest Decomposition of a Complex Signal

Each sample of the signal is a impulse. Decompose the signal to a large number of impulses. Let us cal a complex signal $c[1 \ldots n]$

Finding response to $\delta[t]$ of a system is very simple.

For each sample of c,
Scale & shift the response to delta appropriately.
Add all these up to get response to c.

This is exactly convolution.

Response of a system to delta is called the impulse response, or kernel or filter.

Properties

a) $x[t] * \delta[t] = x[t]$

All pass system.

b) $x[t] * k\delta[t] = kx[t]$

Amplifier (if $k > 1.0$)

Attenuater (if $k < 1.0$)

c) $x[t] * \delta[t+s] = x[t+s]$

Delay System.

Commutative

$$a[t] * b[t] = b[t] * a[t]$$

Associative

$$a[t] * (b[t] * c[t])$$
$$= (a[t] * b[t]) * c[t]$$

(i) Order does not matter.

(ii) Cascading Connection Simplification

$$a[t] \xrightarrow{*} \boxed{h_1} \xrightarrow{*} \boxed{h_2} \xrightarrow{*} y[t]$$

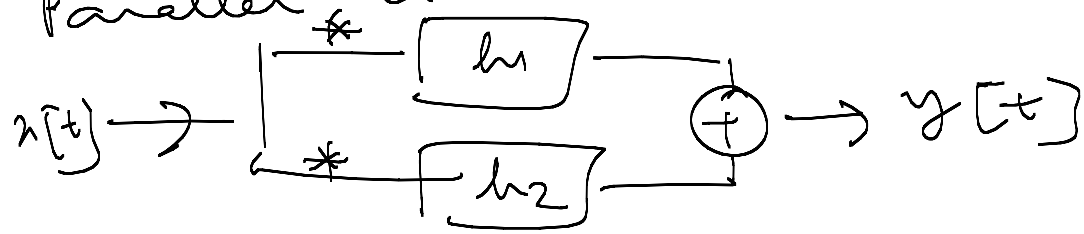
Then

$$a[t] \to \boxed{h_1 * h_2} \to y[t]$$

Distributive

$$a[t] * (b[t] + c[t])$$
$$= a[t] * b[t] + a[t] * c[t]$$

Parallel Connection Simplification



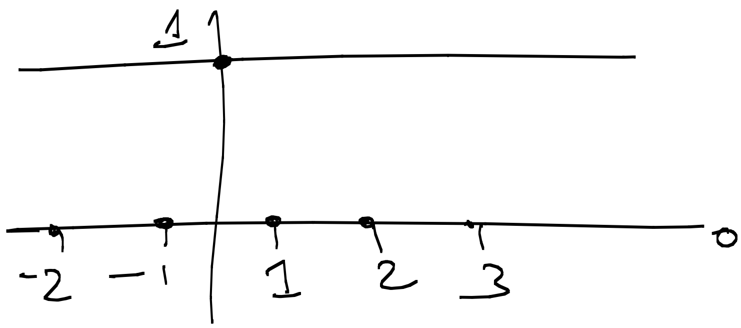
Then

$$x[t] \xrightarrow{*} \boxed{h_1 + h_2} \to y[t]$$

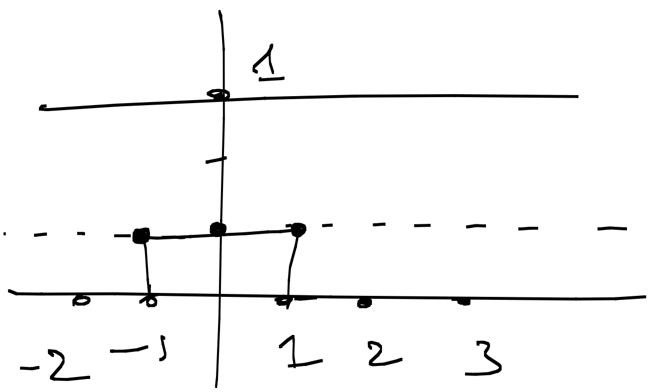
How does this help in filter design?

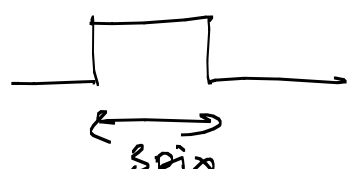
Design a system to blur a signal.
You need to think only about a delta.

δ



Blurred





Having $1/3$ preserves energy (area under the curve)

∴ This is a blurring kernel

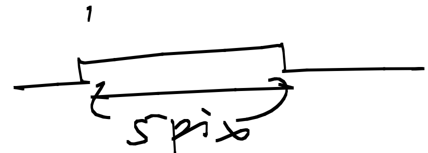
$1/3$	$1/3$	$1/3$

In 2D

$1/9$	$1/9$	$1/9$
$1/9$	$1/9$	$1/9$
$1/9$	$1/9$	$1/9$

What if we want to increase the amount of blur?

1/5	1/5	1/5	1/5	1/5



In 2D, 5x5 with each 1/25.

Now this is also called a low pass filter. Why?

For this we need to know another kind of decomposition of fn called Fourier Transform.

Fourier showed that any complex periodic signal can be expressed as a linear combination of sine & cosine waves. ∴ The sine waves form a linearly independent basis for the set of any periodic signals.

Now the general form of any cos wave is $a\cos(f+p)$

$a \rightarrow$ Amplitude, $f \rightarrow$ freq., $p \rightarrow$ phase

It can be shown that

$$f(t) = \sum_{i=1}^{\infty} a_i \cos(f_i + p_i)$$

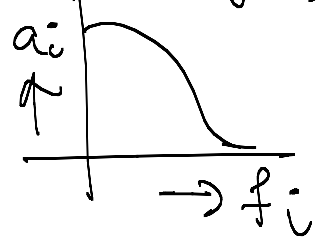
Spatial / time domain representation

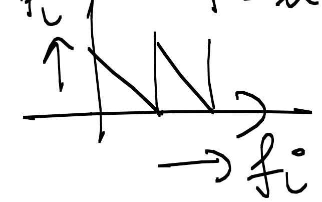
freq. domain representation.

What is this ?

Two plots :

$a = g(f)$ $\quad$ $p = h(f)$





What happens when f is two dimensional.

$$f(x, y) = \sum_{i=1}^{\infty} a_i \cos(f_i, o_i, p_i)$$

Each cos wave now has an orientation too. $\therefore$ $a = g(f, o)$, $p = h(f, o)$

$\therefore$ 2D plots instead of 1D plots.

If f is one direction & o another, perceiving is hard. $\therefore$ A new parametrization.

$$a = g'(k, l)$$
$$p = h'(k, l)$$

where $f = \sqrt{k^2 + l^2}$, $o = \tan^{-1}\left(\frac{l}{k}\right)$

What does this mean?

Bell Shaped

at any (k, l) a wave is represented whose freq is given by the magnitude of the (k, l) vector from $(0, 0)$ & orientation by the orientation of the vector.

Usually Amplitude is the plot we work with most in freq domain. Most of the energy of a signal are in the lower freq. region — provides global appearance. Details are represented in the high freq. region.

Does this mean phase is not important?

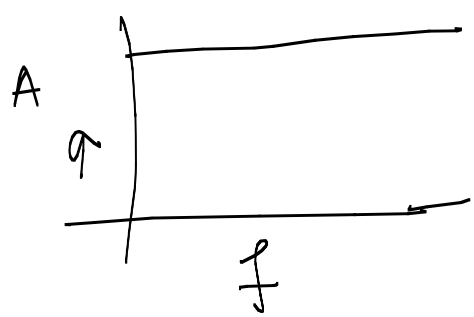
Not really, phase is very important. But what matters is the synchronization of phase. Synchronized rise or fall signifies edges.

∴ easier to deal with in spatial domain.

Coming back to why Low Pass filter?

What would be a freq. response of delta fn?

↳ Amplitude



δ in spatial domain is a constant in freq. domain — equal amount of all freq.

Note the dual property.

A constant in spatial domain is delta in freq. 0 frequency — ∴ delta in freq. domain.

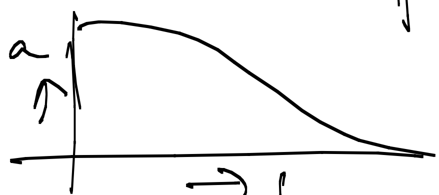
One important property of convolution is

If $a[t] \xrightarrow{F} A[f]$

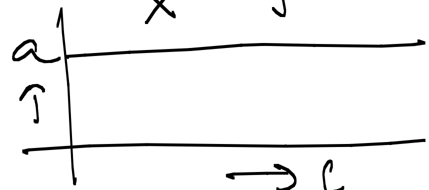
$b[t] \xrightarrow{F} B[f]$

then $a[t] * b[t] \xrightarrow{F} A[f]B[f]$

∴ Why is the name all pass?

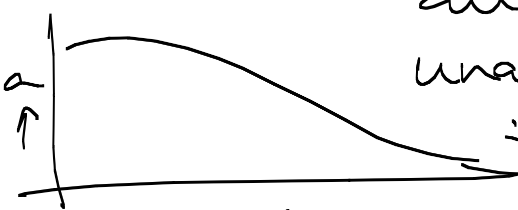
$x[t] \xrightarrow{F}$ 

$\times$ $*$

$\delta[t] \xrightarrow{F}$ 

$\parallel$

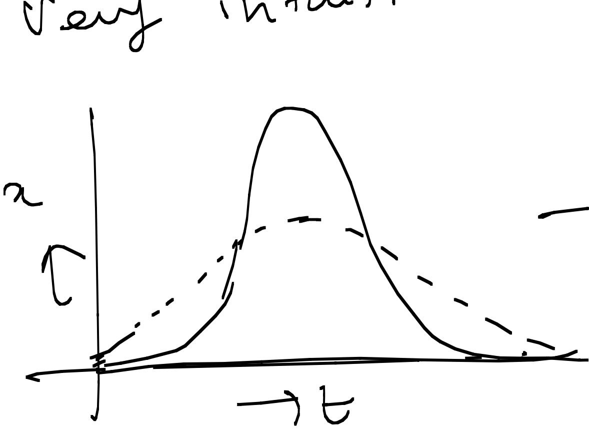
passes all freq. unattenuated

$x[t] \xleftarrow{F^{-1}}$ 

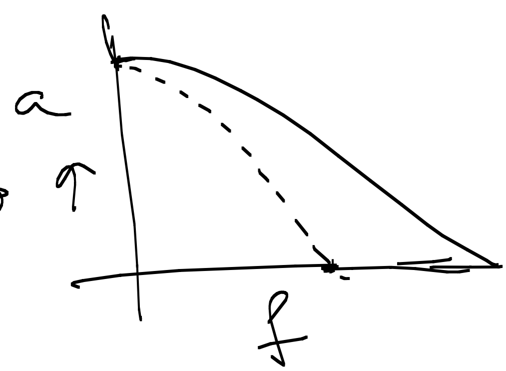
∴ all pass

Another property of Fourier transform

Very intuitive



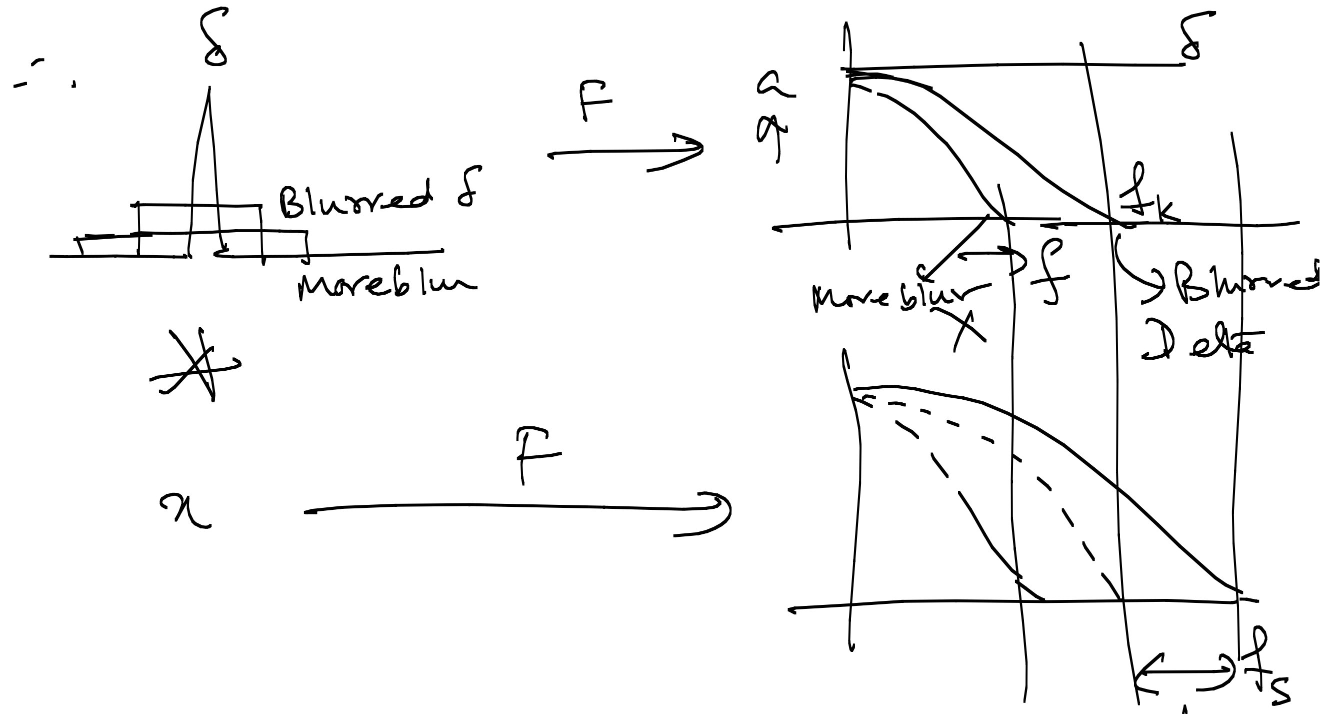
$\xrightarrow{F}$



Increasing width ⟵ Decreasing width

Decreasing width ⟵ Increasing width.

∴ When blurring, instead of δ we have a wider delta



f_s = bandwidth of input
f_k = bandwidth of filter

these high freq. are not passed.

As width of δ is increased, f_k is reduced, and hence more high frequencies are cut off.

Hence LPF.

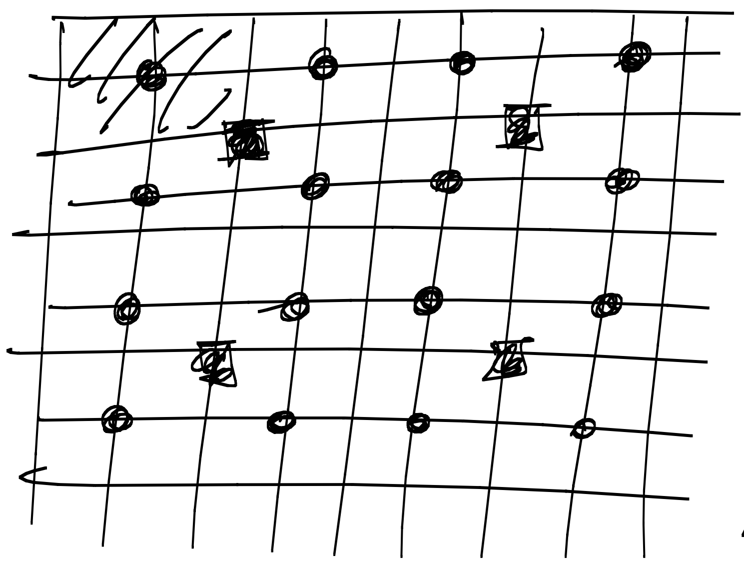
∴ We get a hierarchy by increasing the width. Called Gaussian pyramid.

Issues of Sampling

Note for x, sampling requirement by Nyquist criteria is $2f_s$. But the LPF image has a lower sampling

requirement of $2f_k$ since $f_k < f_s$.
This is utilized when building the gaussian pyramid.

Instead of increasing the kernel size, the same kernel is applied iteratively. Example.



$G_1 = 8 \times 8$

$G_2 = 4 \times 4$

by applying 2×2 kernel since sampling requirement has gone down due to blurring 4×4 is sufficient.

Now apply the same 2×2 kernel to this 4×4 image to generate a 2×2 image. This is effectively applying a 4×4 kernel to G_1. $\therefore$ Effectively widening the kernel.
Since convolution is associative this will work.

$$(G_1 * h) * h = G_1 * \underbrace{(h * h)}_{\text{wider kernel}}$$

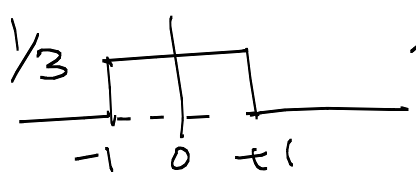
But sampling demand is exploited to get smaller images.

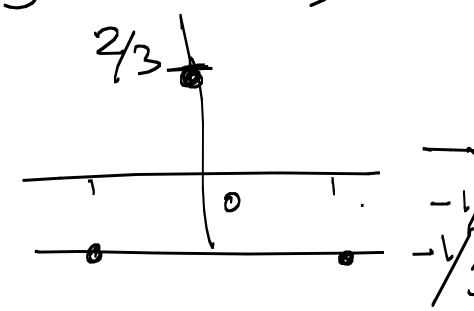
How can we design a high pass filter?

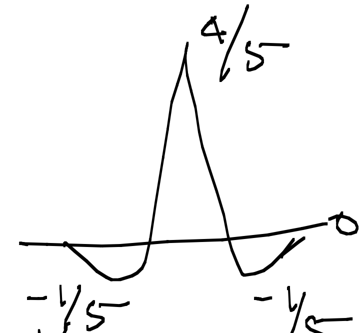
$$x[t] - x[t] * h[t]$$

$$= x[t] * \delta[t] - x[t] * h[t]$$

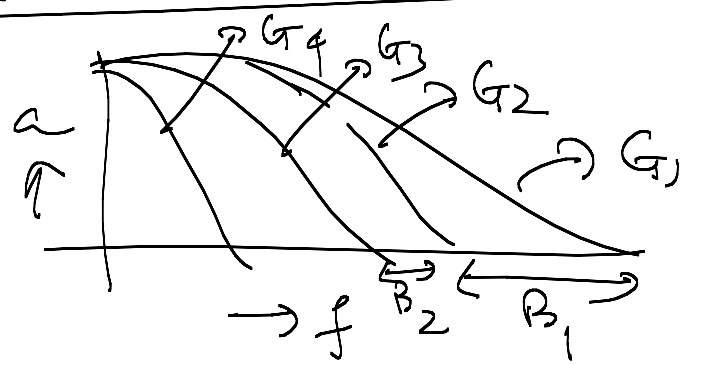
$$= x[t] * (\delta[t] - h[t])$$







Band Pass Filters



$B_1 = G_1 - G_2$

$B_2 = G_2 - G_3$

$\vdots$

$B_{n-1} = G_{n-1} - G_n$

Called Laplacian Pyramid.

2D separability

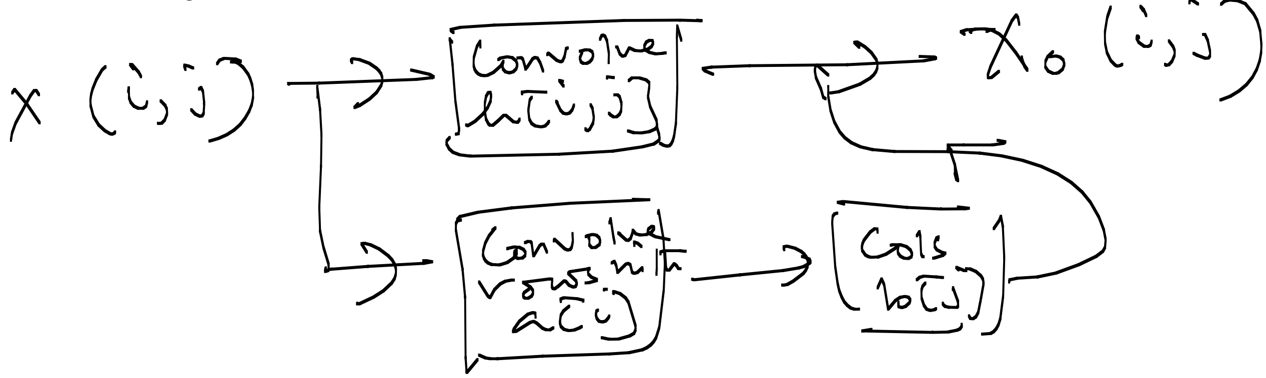
Let $h[i,j]$ be a kernel / filter. If it can be broken into two 1D filter, $a[i]$, $b[j]$ s.t $h[i,j] = a[i] \times b[j]$

Then $h[i,j]$ is separable.

Eg. $a[i] = \boxed{1/3 \,|\, 1/3 \,|\, 1/3}$ $\quad b[j] = \boxed{1/3 \,|\, 1/3 \,|\, 1/3}$

$\therefore h[i,j] = 1/9$ for $0 \le i,j \le 3$

Why useful?



If with $h[i,j]$ $\quad 0 \le i \le p$, $\quad 0 \le j \le q$

Each pix take pq prod + pq sum $= 2pq$ $\quad x = m \times n$ img.

For mn pixels

$2pq\, mn$ operation.

If separable

For a, $2p\, mn$ operation

b, $2q\, mn$ "

$\therefore$ Total $2mn(p+q)$ operations.

Why does it work?

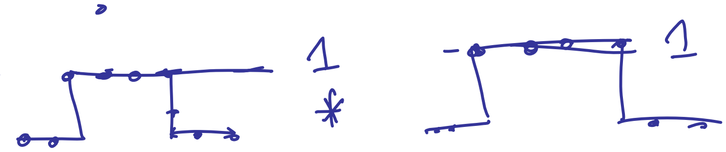
Check $h[i,j] = a[i] * b[j]$

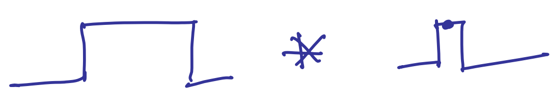
$$x * h$$
$$= x * (a * b)$$
$$= (x * a) * b.$$

Correlation

How much is a signal similar to another?

$$\sum_{i=1}^{4} 1 \times 1 = 4$$

Say 

But say 

$$1\times 1 + 1\times 0 = 1$$

$\therefore$ Correlation is a measure of how closely is the signal matches another.

$\therefore$ It is a convolution with target signal. Think of target signal as the impulse response of the system.

Properties

a) Does not matter if signal is +ve or -ve. Correlation is always +ve.

b) Symmetric on both sides of peak, (even if target is not).

c) Width of the peak is twice the target.

If with itself, it is called Auto-correlation.