Outline

• Greedy local search
• Random Walk Strategies; e.g. walksat
• Hybrid of local search and Inference
  • The impact of constraint propagation
  • Local search on the cycle-cutset
Example: 8-queen problem
Example: 8-queen problem
Example: 8-queen problem

Figure 6.9 A two-step solution using min-conflicts for an 8-queens problem. At each stage, a queen is chosen for reassignment in its column. The number of conflicts (in this case, the number of attacking queens) is shown in each square. The algorithm moves the queen to the min-conflicts square, breaking ties randomly.
Hopfield networks

Find an assignment of 0, 1 to the unit that minimize the energy.

\[ E = \sum_{i < j} w_{ij} s_i s_j + \sum_i b_i s_i \]

\[ S(s_i = 1) - E(S_i = 0) = b_i + \sum_i w_{ij} s_j \]
Greedy local search

• Choose a full assignment and iteratively improve it towards a solution
• Requires a cost function: number of unsatisfied constraints or clauses.
• Neural networks use energy minimization
• Drawback: local minimas
• Remedy: introduce a random element
• Cannot decide inconsistency
function MIN-CONFLICTS(csp, max_steps) returns a solution or failure
inputs: csp, a constraint satisfaction problem
        max_steps, the number of steps allowed before giving up

        current ← an initial complete assignment for csp
for i = 1 to max_steps do
    if current is a solution for csp then return current
    var ← a randomly chosen conflicted variable from csp.VARIABLES
    value ← the value v for var that minimizes CONFLICTS(var, v, current, csp)
    set var = value in current
return failure

Figure 6.8 The MIN-CONFLICTS algorithm for solving CSPs by local search. The initial state may be chosen randomly or by a greedy assignment process that chooses a minimal-conflict value for each variable in turn. The CONFLICTS function counts the number of constraints violated by a particular value, given the rest of the current assignment.
Algorithm Stochastic Local search (SLS)

```
Procedure SLS
Input: A constraint network $\mathcal{R} = (X, D, C)$, number of tries MAX_TRIES. A cost function.
Output: A solution iff the problem is consistent, ”false” otherwise.

1. for i=1 to MAX_TRIES
   - initialization: let $\bar{a} = (a_1, ..., a_n)$ be a random initial assignment to all variables.
   - repeat
     (a) if $\bar{a}$ is consistent, return $\bar{a}$ as a solution.
     (b) else let $Y = \{ < x_i, a'_i > \}$ be the set of variable-value pairs that when $x_i$ is assigned $a'_i$, give a maximum improvement in the cost of the assignment; pick a pair $< x_i, a'_i > \in Y$.
     $\bar{a} \leftarrow (a_1, ..., a_{i-1}, a'_i, a_{i+1}, ..., a_n)$ (just flip $a_i$ to $a'_i$).
   - until the current assignment cannot be improved.

2. endfor
3. return false
```

GSAT: SLS when the cost is the number of unsatisfied clauses.
Example of GSAT: CNF

**Example 7.1** Consider the formula $\varphi = \{ (\neg C) (\neg A \vee \neg B \vee C) (\neg A \vee D \vee E) (\neg B \vee \neg C) \}$. Assume that in the initial assignment all variables are assigned the value “1”. This
Example of GSAT: CNF

**Example 7.1** Consider the formula $\varphi = \{(\neg C)(\neg A \vee \neg B \vee C)(\neg A \vee D \vee E)(\neg B \vee \neg C)\}$. Assume that in the initial assignment all variables are assigned the value "1". This assignment violates two clauses, the first and the last, so the cost is 2. Next we see that flipping A, E or D will not remove any inconsistency. Flipping C to "0" will satisfy the two violated clauses but will violate the clause $(\neg A \vee \neg B \vee C)$, yielding a cost of 1. Flipping $B$ to $\neg B$ will remove one inconsistency and has a cost of 1 as well. If we flip $C$ to $\neg C$, and subsequently flipping $B$ to $\neg B$ yields a cost of 0 — and a solution. □

- **Example:** $z$ divides $y, x, t$
- $z = \{2, 3, 5\}$, $x, y = \{2, 3, 4\}$, $t = \{2, 5, 6\}$
Local Minima
Heuristics for improving local search

• **Plateau search**: at local minima continue search sideways.

• **Constraint weighting**: use weighted cost function
  - The cost $C_i$ is 1 if $C_i$ is violated. At local minima increase the weights of violating constraints. Choose $(V,v)$ pair that leads to largest reduction of $F$
    $$F(\bar{a}) = \sum w_i C_i(\bar{a})$$

• **Tabu search**:
  - prevent backwards moves by keeping a list of assigned variable-values. Tie-breaking rule may be conditioned on historic information: select the value that was flipped least recently

• **Automating Max-flips**:
  - Based on experimenting with a class of problems
  - Given a progress in the cost function, allow the same number of flips used up to current progress.
Random walk strategies

• Combine random walk with greediness
  • At each step:
    • choose randomly an unsatisfied clause.
    • with probability $p$ flip a random variable in the clause, with $(1-p)$ do a greedy step minimizing the breakout value: the number of new constraints that are unsatisfied
Figure 7.2: Algorithm WalkSAT

Procedure WalkSAT
Input: A network $\mathcal{R} = (X, D, C)$, number of flips MAX_FLIPS, MAX_TRIES, probability $p$.
Output: True iff the problem is consistent, false otherwise.

1. For $i = 1$ to MAX_TRIES do

2. Compare best assignment with $\bar{a}$ and retain the best.
   (a) start with a random initial assignment $\bar{a}$.
   (b) for $i = 1$ to MAX_FLIPS
       ● if $\bar{a}$ is a solution, return true and $\bar{a}$.
       ● else,
         i. pick a violated constraint $C$, randomly
         ii. choose with probability $p$ a variable-value pair $< x, a' >$ for $x \in \text{scope}(C)$, or, with probability $1 - p$, choose a variable-value pair $< x, a' >$ that minimizes the number of new constraints that break when the value of $x$ is changed to $a'$, (minus 1 if the current constraint is satisfied).
         iii. Change $x$’s value to $a'$.
   endfor
3. endfor
4. return false and the best current assignment.
Example of walkSAT:

start with assignment of true to all vars

Example 7.2 Following our earlier example 7.1.1, we will first select an unsatisfied clause, such as \((\neg B \lor \neg C)\), and then select a variable.

\[\neg C, (\neg A \lor \neg B \lor C)(\neg A \lor D \lor E)(\neg B \lor \neg C)\]
Example of walkSAT: start with assignment of true to all vars

Example 7.2 Following our earlier example 7.1.1, we will first select an unsatisfied clause, such as \((\neg B \lor \neg C)\), and then select a variable. If we try to minimize the number of additional constraints that would be broken, we will select \(B\) and flip its value. Subsequently, the only unsatisfied clause is \(\neg C\) which is selected and flipped.

\[
(\neg C), (\neg A \lor \neg B \lor C)(\neg A \lor D \lor E)(\neg B \lor \neg C)
\]
Simulated Annealing
(Kirkpatric, Gellat and Vecchi (1983))

• Pick randomly a variable and a value and compute delta: the change in the cost function when the variable is flipped to the value.
• If change improves execute it,
• Otherwise it is executed with probability $e^{\delta / T}$ where $T$ is a temperature parameter.
• The algorithm will converge if $T$ is reduced gradually.
Properties of local search

- Guarantee to terminate at local minima
- Random walk on 2-sat is guaranteed to converge with probability 1 after $N^2$ steps, when N is the number of variables.
- Proof:
  - A random assignment is on the average $N^2$ flips away from a satisfying assignment.
  - There is at least $\frac{1}{2}$ chance that a flip of a 2-clause will reduce the distance to a given satisfying assignment by 1 (because the satisfying assignment assigns true to at least one literal in a randomly picked unsatisfied 2-clause, so at least 50% chance a flip will satisfy it).
  - Random walk will cover this distance in $N^2$ steps on the average.
- Analysis breaks for 3-SAT
- Empirical evaluation shows good performance compared with complete algorithms (see chapter and numerous papers)
Comparing various styles of SLS

The random walk strategy (i.e., GSAT +walk) augments GSAT as follows:
• with a probability $p$, pick a variable occurring in an unsatisfied clause and flip its truth value. With probability $1 - p$ do a regular greedy step.

The random noise strategy is the same except the variable can be picked from any clause.

Both random walk and random noise differ from WalkSAT in a subtle way.

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<th>flips</th>
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Table 7.1: Comparing noise strategies on hard random 3CNF instances.
Comparing DPLL and local search on circuit synthesis problems

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<th>WSAT time</th>
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<td>0.001</td>
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Table 7.2: Comparing complete DPLL method (DP) with local search strategies on circuit synthesis problems. (Timings in seconds.)
Outline

• Greedy local search
• Random Walk Strategies; e.g. walksat
• Hybrid of local search and Inference
  • The impact of constraint propagation
  • Local search on the cycle-cutset
Hybrids of local search and inference

• We can use exact hybrids of search+inference and replace search by SLS (Kask and Dechter 1996)
  • Good when cutset is small

• The effect of preprocessing by constraint propagation on SLS (Kask and Dechter 1995)
  • Great improvement on structured problems
  • Not so much on uniform problems
SLS and Local Consistency

- **Structured** (hierarchical 3SAT cluster structures) vs. (uniform) random.

**Basic scheme:**
- Apply preprocessing (resolution, path consistency)
- Run SLS
- Compare against SLS alone

What can we say about local search when we have the minimal network?
SLS and Local Consistency


[Kask and Dechter, Ijcai 1995]
SLS and SLS on structured problems

<table>
<thead>
<tr>
<th>C/cluster</th>
<th>Before Resolution</th>
<th>After Resolution</th>
<th>DP</th>
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<td>Flips</td>
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<td>4.5K</td>
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<td>0.71</td>
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<td>100</td>
<td>1.03</td>
<td>8.4K</td>
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<td>100</td>
<td>1.54</td>
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<tr>
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<td>-</td>
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Table 1: Bound-3 Resolution and GSAT

SLS and Local Consistency

<table>
<thead>
<tr>
<th>C</th>
<th>Solvable</th>
<th>Algorithm</th>
<th>Solved</th>
<th>Tries</th>
<th>Flips</th>
<th>PPC Time</th>
<th>Total Time</th>
<th>BJ-DVO</th>
</tr>
</thead>
<tbody>
<tr>
<td>265</td>
<td>88.5%</td>
<td>GSAT</td>
<td>139</td>
<td>336</td>
<td>147K</td>
<td>0 sec</td>
<td>36 sec</td>
<td>19 min</td>
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<tr>
<td></td>
<td></td>
<td>PPC + GSAT</td>
<td>152</td>
<td>292</td>
<td>140K</td>
<td>8 sec</td>
<td>66 sec</td>
<td></td>
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<tr>
<td>270</td>
<td>66%</td>
<td>GSAT</td>
<td>78</td>
<td>406</td>
<td>191K</td>
<td>0 sec</td>
<td>45 sec</td>
<td>33 min</td>
</tr>
<tr>
<td></td>
<td></td>
<td>PPC + GSAT</td>
<td>83</td>
<td>381</td>
<td>195K</td>
<td>14 sec</td>
<td>92 sec</td>
<td></td>
</tr>
</tbody>
</table>

N=100, K=8, T=32/64, 200 instances, MaxFlips = 512K

<table>
<thead>
<tr>
<th>C</th>
<th>Solvable</th>
<th>Algorithm</th>
<th>Solved</th>
<th>Tries</th>
<th>Flips</th>
<th>PPC Time</th>
<th>Total Time</th>
<th>BJ-DVO</th>
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<tbody>
<tr>
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<td>56</td>
<td>276</td>
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<tr>
<td></td>
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<td>53K</td>
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<td>89 sec</td>
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N=30, K=64, T=2048/4096, 100 instances, MaxFlips = 128K, C_{crit}=180

Table 2: Partial Path Consistency and GSAT

<table>
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<tr>
<th>Algorithm</th>
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<th>Tries</th>
<th>Flips</th>
<th>BR-3 Time</th>
<th>Total Time</th>
</tr>
</thead>
<tbody>
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<td>63</td>
<td>176K</td>
<td>0 sec</td>
<td>15.3 sec</td>
</tr>
<tr>
<td>BR-3 + GSAT</td>
<td>31</td>
<td>45</td>
<td>125K</td>
<td>0.3 sec</td>
<td>15.0 sec</td>
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</table>

Uniform random 3SAT, N=600, C=2550, 100 instances, MaxFlips = 512K

Table 3: Bound-3 Resolution and GSAT
SLS and Local Consistency

Summary:

• For structured problems, enforcing local consistency will improve SLS
• For uniform CSPs, enforcing local consistency is not cost effective: performance of SLS is improved, but not enough to compensate for the preprocessing cost.
Outline

• Greedy local search
• Random Walk Strategies; e.g. walksat
• Hybrid of local search and Inference
  • The impact of constraint propagation
  • Local search on the cycle-cutset
SLS and Cutset Conditioning

**Background:**
- Tree algorithm is tractable for trees.
- Networks with bounded width are tractable*.

**Basic Scheme:**
- Identify a cutset such that width is reduced to desired value.
- Use search with cutset conditioning.
Local search on Cycle-cutset

Tree variables $\mathbf{X}$

Cutset $\mathbf{Y}$

$x_i = \{0, 1\}$

$x_j = \{0, 1\}$
Tree Algorithm: minimizing the cost of a tree-like subnetwork:
where $R_{z_i, z_j}(a_i, a_j)$ is the constraint between $z_i$ and $z_j$ and is either 0 if $(a_i, a_j) \in R_{z_i, z_j}$ or 1, otherwise.

Input: An arc consistent network $\mathcal{R} = (X, D, C)$. Variables $X$ partitioned into cycle cutset $Y$ and tree variables $Z$, $X = Z \cup Y$. An assignment $Y = \bar{y}$.

Output: An assignment $Z = \bar{z}$ that minimizes the number of violated constraints of the entire network when $Y = \bar{y}$.

Initialization: For any value $\bar{y}[i]$ of any cutset variable $y_i$, the cost $C_{y_i}(\bar{y}[i], \bar{y})$ is 0.

1. Going from leaves to root on the tree,

   (a) for every variable, $z_i$ and any value $a_i \in D_{z_i}$, compute,

   $$C_{z_i}(a_i, \bar{y}) = \sum_{\{z_j | z_j \text{ child of } z_i\}} \min_{a_j \in D_{z_j}} (C_{z_j}(a_j, \bar{y}) + R_{z_i, z_j}(a_i, a_j))$$

   (b) endfor
2. Compute, going from root to leaves, new assignment for every tree variable $z_i$:

   (a) for a tree variable $z_i$, let $D_{z_i}$ be its consistent values with $v_{p_i}$ the value assigned to its parent $p_i$, compute

   \[ a_i \leftarrow \arg \min_{a_i \in D_{z_i}} (C_{z_i}(a_i, y) + R_{z_i, p_i}(a_i, v_{p_i})) \]

   (b) endfor

3. return $(<z_1, a_1>, ..., <z_k, a_k>)$. 
GSAT with cycle-cutset (Kask and Dechter, 1996)

**Input:** a CSP, a partition of the variables into **cycle-cutset** and **tree variables**

**Output:** an assignment to all the variables

**Within each try:**
Generate a random initial assignment, and then alternate between the two steps:

1. Run **Tree algorithm** (arc-consistency+assignment) on the problem with fixed values of cutset variables.
2. Run GSAT on the problem with fixed values of tree variables.
Theorem 7.1 The Tree Algorithm in Figure 7.4 is guaranteed to find an assignment that minimizes the number of violated constraints in every tree-like subnetwork, conditioned on the cutset values.
Results GSAT with Cycle-Cutset
(Kask and Dechter, 1996)
SLS and Cutset Conditioning

**Summary:**

- A new combined algorithm of SLS and inference based on cutset conditioning
- Empirical evaluation on random CSPs
- SLS combined with the tree algorithm is superior to pure SLS when the cutset is small

Outline

• Greedy local search
• Random Walk Strategies; e.g. walksat
• Hybrid of local search and Inference
  • The impact of constraint propagation
  • Local search on the cycle-cutset