CONSTRAINT Networks
Chapters 1-2

Compsci-275

Winter 2016

Class Information

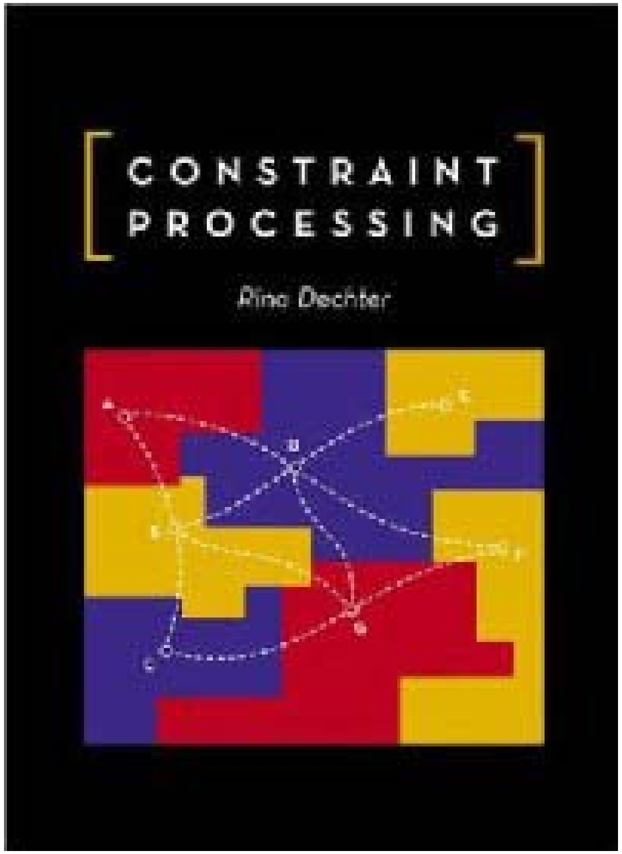
- Instructor: Rina Dechter
- Lectures: Monay & Wednesday
- Time: 11:00 - 12:20 pm
- Discussion (optional): Wednesdays 12:30-1:20
- Class page: http://www.ics.uci.edu/~dechter/courses/ics-275a/spring-2014/

Text book (required)

Rina Dechter,

[Constraint Processing](#),

Morgan Kaufmann



Outline

- ✓ Motivation, applications, history
- ✓ CSP: Definition, and simple modeling examples
- ✓ Mathematical concepts (relations, graphs)
- ✓ Representing constraints
- ✓ Constraint graphs
- ✓ The binary Constraint Networks properties

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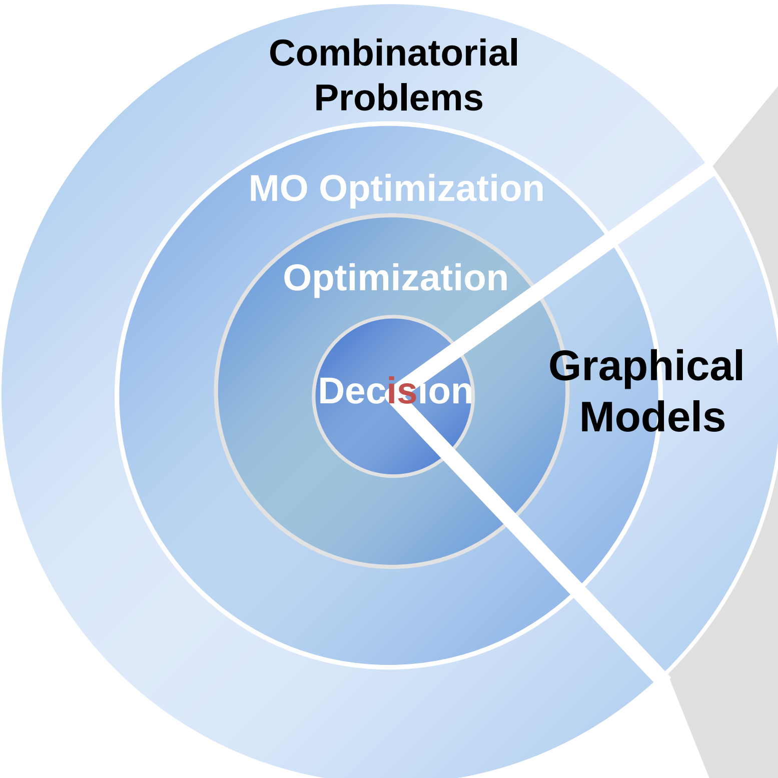


Graphical Models

Those problems that can be expressed as:

A set of variables

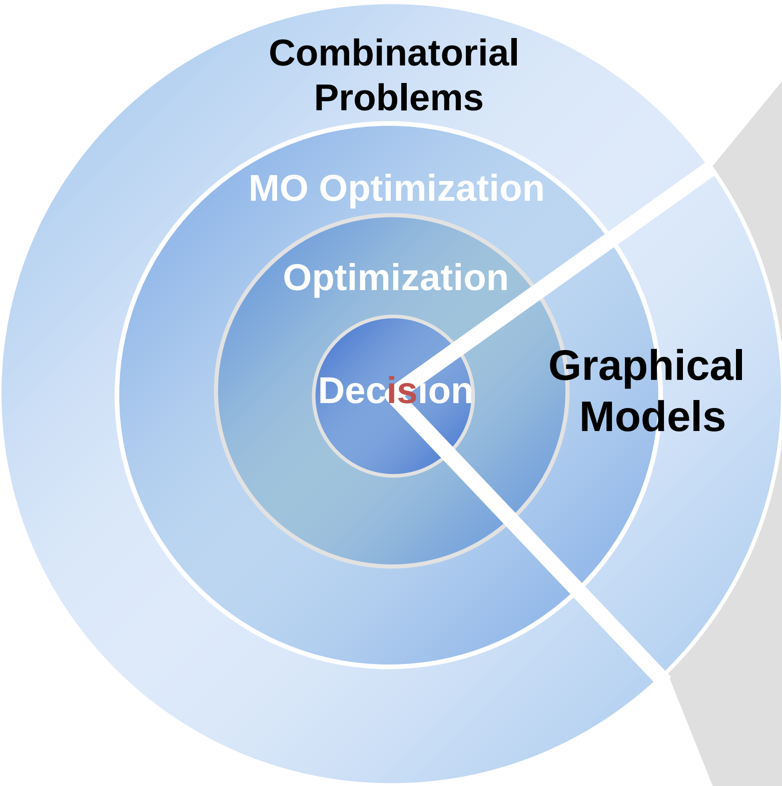
Each variable takes its values from a finite set of domain values

A set of local functions

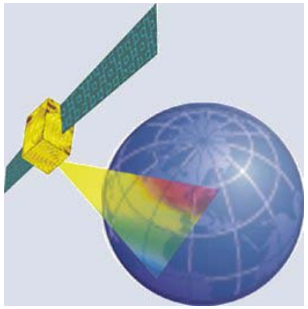
Main advantage:

They provide unifying algorithms:

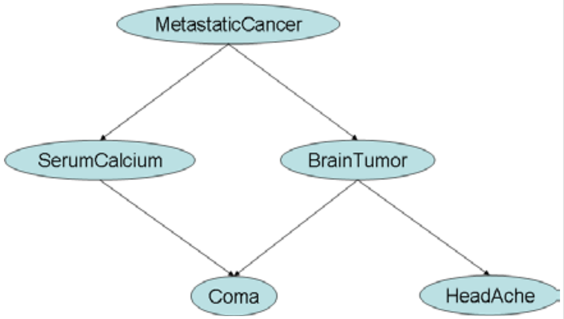
- Search
- Complete Inference
- Incomplete Inference



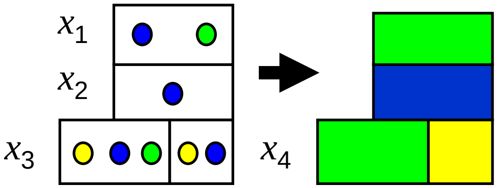
Many Examples



EOS Scheduling



Bayesian Networks



Graph Coloring

Timetabling

… and many others.

Example: student course selection

- **Context**: You are a senior in college
- **Problem**: You need to register in 4 courses for the Spring semester
- **Possibilities**: Many courses offered in Math, CSE, EE, CBA, etc.
- **Constraints**: restrict the choices you can make
  - Courses have prerequisites you have/don't have Courses/instructors you like/dislike
  - Courses are scheduled at the same time
  - In CE: 4 courses from 5 tracks such as at least 3 tracks are covered
- **You have choices, but are restricted by constraints**
  - Make the right decisions!!
  - ICS Graduate program

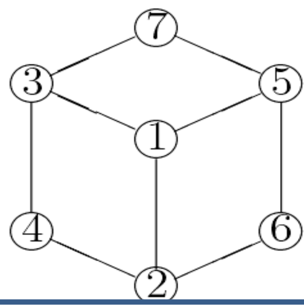
Student course selection (continued)

- **Given**
  - A set of variables: 4 courses at your college
  - For each variable, a set of choices (values): the available classes.
  - A set of constraints that restrict the combinations of values the variables can take at the same time
- **Questions**
  - Does a solution exist? (classical decision problem)
  - How many solutions exists? (counting)
  - How two or more solutions differ?
  - Which solution is preferable?
  - etc.

The field of Constraint Programming

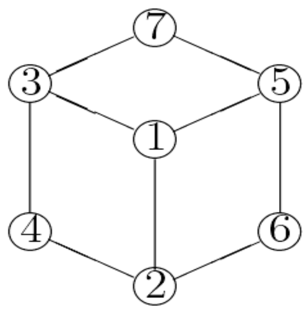
- **How did it started:**
  - Artificial Intelligence (vision)
  - Programming Languages (Logic Programming),
  - Databases (deductive, relational)
  - Logic-based languages (propositional logic)
  - SATisfiability
- **Related areas:**
  - Hardware and software verification
  - Operation Research (Integer Programming)
  - Answer set programming
- **Graphical Models; deterministic**

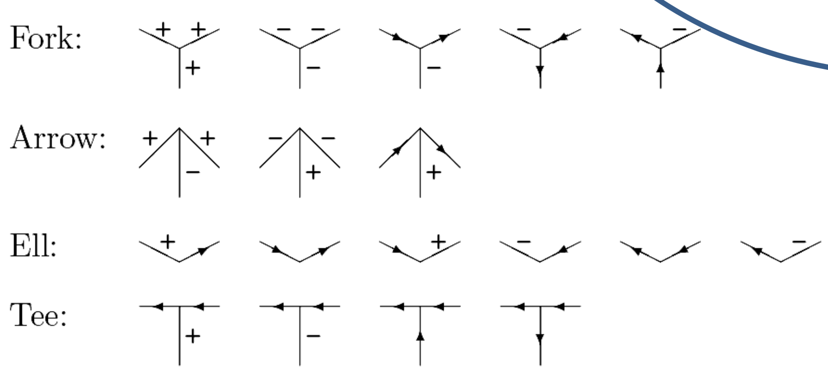
Scene labeling constraint network



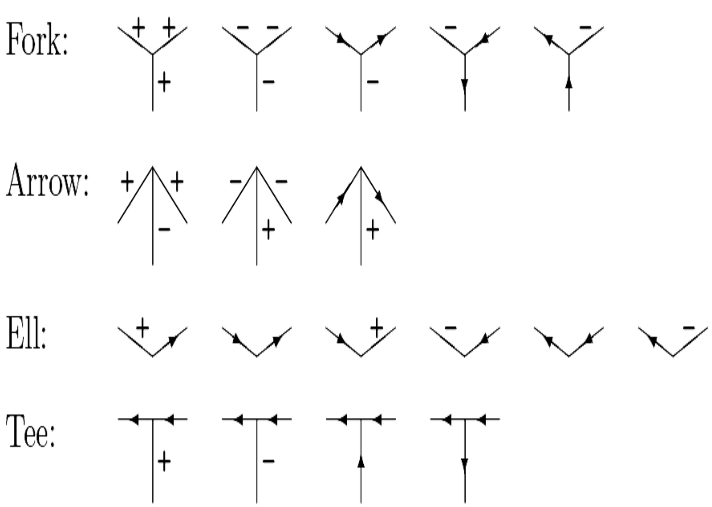
R

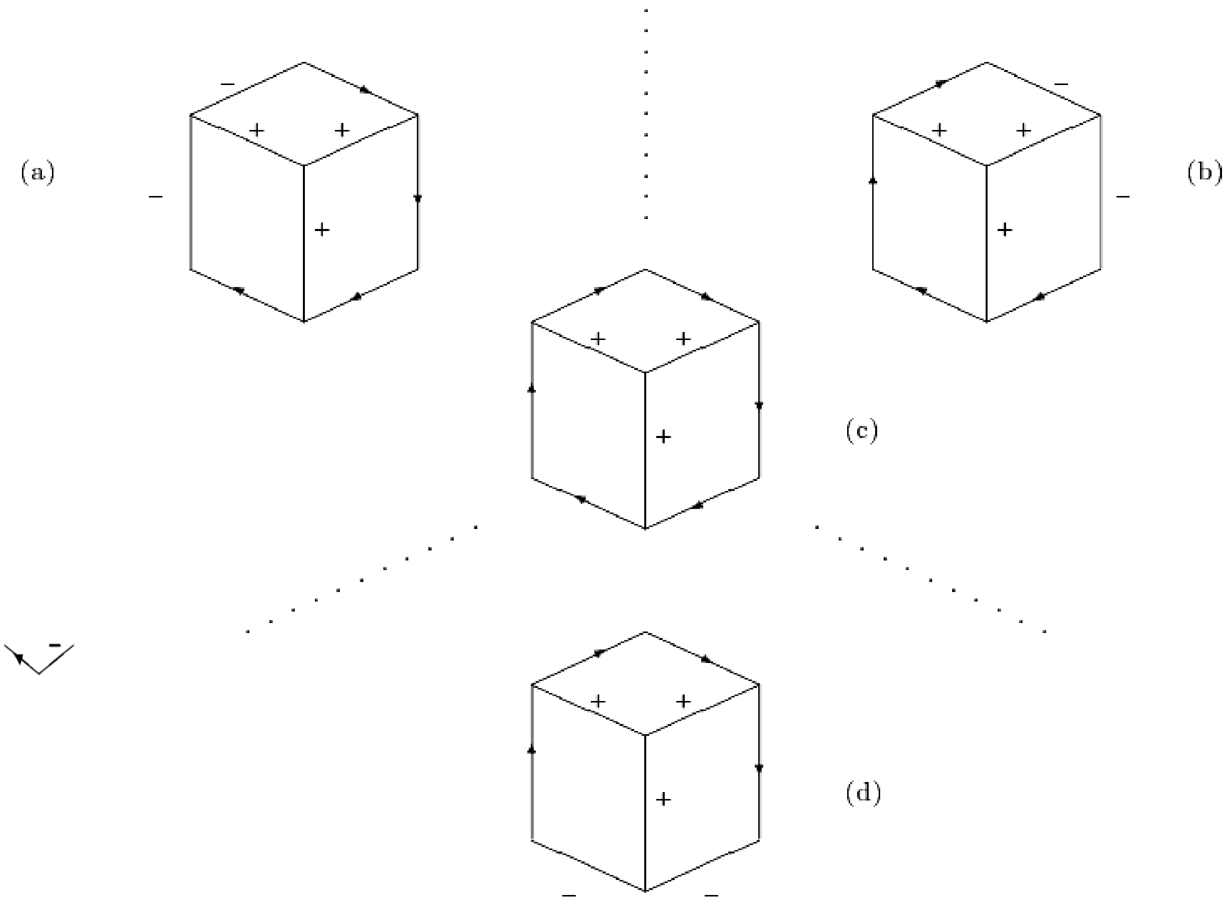
Scene labeling constraint network





3-dimentional interpretation of 2-dimentional drawings





The field of Constraint Programming

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Applications

- Radio resource management (RRM)
- Databases (computing joins, view updates)
- Temporal and spatial reasoning
- Planning, scheduling, resource allocation
- Design and configuration
- Graphics, visualization, interfaces
- Hardware verification and software engineering
- HC Interaction and decision support
- Molecular biology
- Robotics, machine vision and computational linguistics
- Transportation
- Qualitative and diagnostic reasoning

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Constraint Networks

A

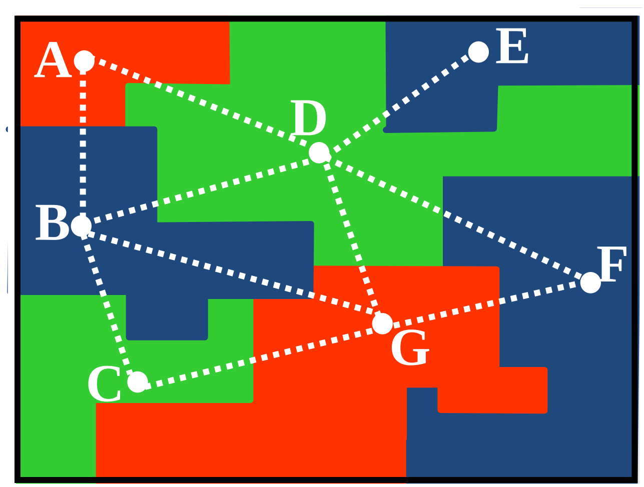
Example: map coloring

Variables - countries (A,B,C,etc.)

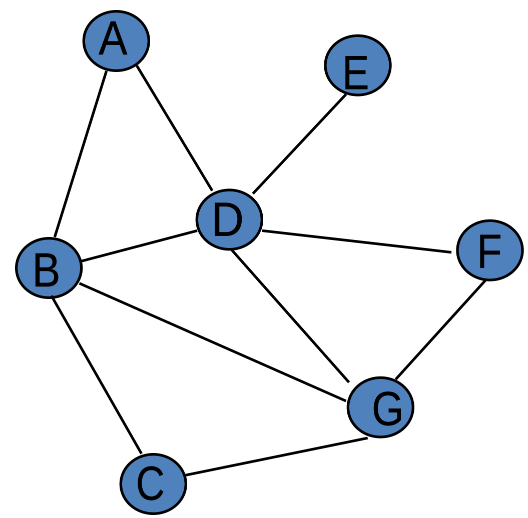
Values - colors (red, green, blue)

Constraints: $A \neq B$, $A \neq D$, $D \neq E$, *etc.*

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red



Constraint graph



Constraint Satisfaction Tasks

Example: map coloring

Variables - countries (A,B,C,etc.)

Values - colors (e.g., red, green, yellow)

Constraints: $A \neq B$, $A \neq D$, $D \neq E$, *etc.*

A	B	C	D	E...
red	green	red	green	blue
red	blue	green	green	blue
...	...	...	...	green
...	...	...	...	red
red	blue	red	green	red

Are the constraints consistent?

Find a solution, find all solutions

Count all solutions

Find a good solution

Information as Constraints

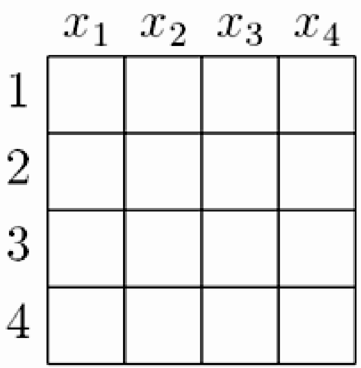
- I have to finish my class in 50 minutes
- 180 degrees in a triangle
- Memory in our computer is limited
- The four nucleotides that makes up a DNA only combine in a particular sequence
- Sentences in English must obey the rules of syntax
- Susan cannot be married to both John and Bill
- Alexander the Great died in 333 B.C.

Constraint Network; Definition

- A constraint network is: **R=*(X,D,C)***
  - **X variables** $X = \{X_1, \ldots, X_n\}$
  - **D domain** $D = \{D_1, \ldots, D_n\}, D_i = \{v_1, \ldots v_k\}$
  - **C constraints** $C = \{C_1, \ldots C_t\},,, C_i = (S_i, R_i)$
  - ***R* expresses allowed tuples over scopes**
- **A solution** is an assignment to all variables that satisfies all constraints (join of all relations).
- ***Tasks:*** *consistency?, one or all solutions, counting, optimization*

The N-queens problem

The network has four variables, all with domains D_i = {1, 2, 3, 4}.
(a) The labeled chess board. (b) The constraints between variables.



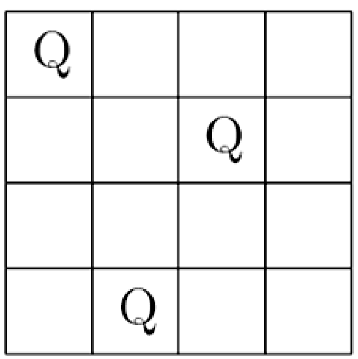
(a)

$$
\begin{aligned}
R_{12} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\} \\
R_{13} &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\} \\
R_{14} &= \{(1,2), (1,3), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4) \\
&\quad\ (4,2), (4,3)\} \\
R_{23} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\} \\
R_{24} &= \{(1,2), (1,4), (2,1), (2,3), (3,2), (3,4), (4,1), (4,3)\} \\
R_{34} &= \{(1,3), (1,4), (2,4), (3,1), (4,1), (4,2)\}
\end{aligned}
$$

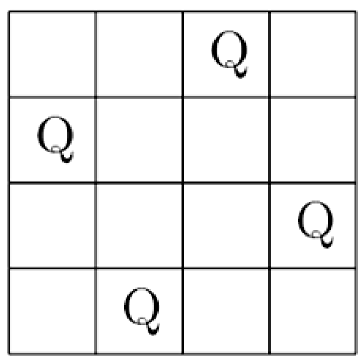
(b)

A solution and a partial consistent tuple

Not all consistent instantiations are part of a solution:
(a) A consistent instantiation that is not part of a solution.
(b) The placement of the queens corresponding to the solution (2, 4, 1,3).
c) The placement of the queens corresponding to the solution (3, 1, 4, 2).



(a)



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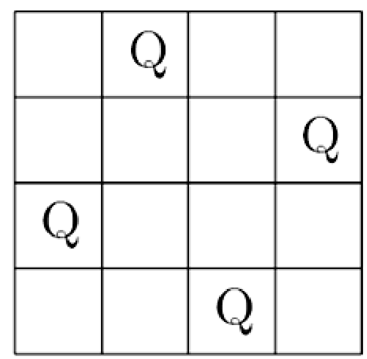
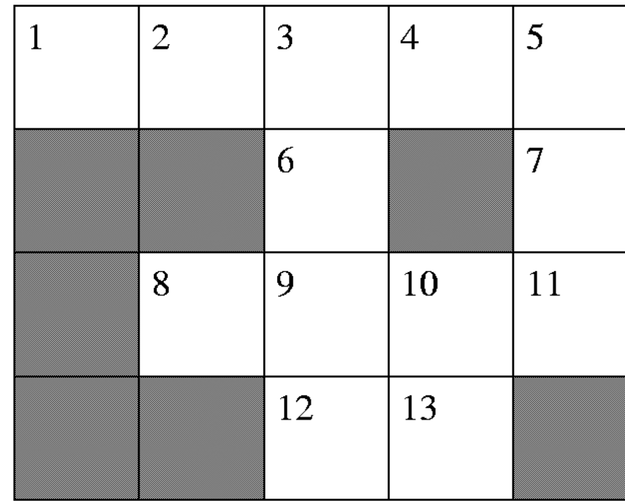


(c)

Example: Crossword puzzle

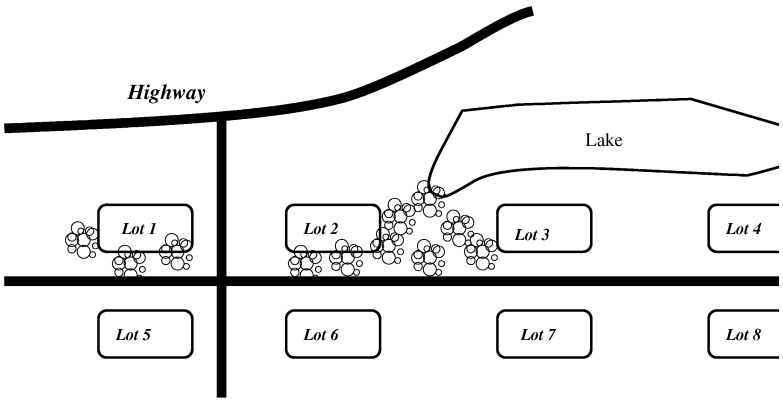
- Variables: $x_1, \ldots, x_{13}$
- Domains: letters
- Constraints: words from



{HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US}

Configuration and Design

- Want to build: recreation area, apartment complex, a cluster of 50 single-family houses, cemetery, and a dump
  - Recreation area near lake
  - Steep slopes avoided except for recreation area
  - Poor soil avoided for developments
  - Highway far from apartments, houses and recreation
  - Dump not visible from apartments, houses and lake
  - Lots 3 and 4 have poor soil
  - Lots 3, 4, 7, 8 are on steep slopes
  - Lots 2, 3, 4 are near lake
  - Lots 1, 2 are near highway



Example: Sudoku
(constraint propagation)

Constraint propagation

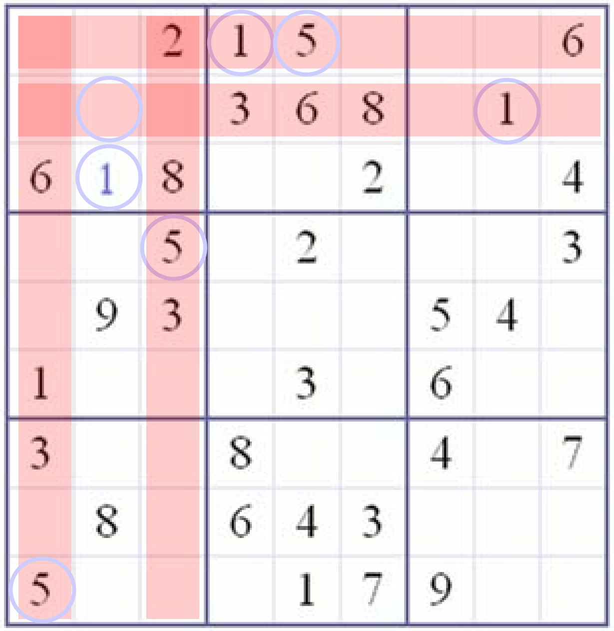
		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	2 ~~3~~ ~~4~~ ~~6~~ **2**
		9			4	5	8	1
			3		2	9		

- **Variables**: 81 slots
- **Domains** = {1,2,3,4,5,6,7,8,9}
- **Constraints**:
  - 27 not-equal

Each row, column and major block must be alldifferent

"Well posed" if it has unique solution: 27 constraints

Sudoku (inference)



Each row, column and major block must be alldifferent

“Well posed” if it has unique solution

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- ✓ **Mathematical concepts (relations, graphs)**
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Mathematical background

- Sets, domains, tuples
- Relations
- Operations on relations
- Graphs
- Complexity

Two Representations of a relation:
R = {(black, coffee), (black, tea), (green, tea)}.

Variables: Drink, color

x_1	x_2
black	coffee
black	tea
green	tea

(a) table

Two Representations of a relation:
R = {(black, coffee), (black, tea), (green, tea)}.

Variables: Drink, color

x_1	x_2
black	coffee
black	tea
green	tea

(a) table

$$x_1 \quad \begin{array}{c} \\ \text{black} \\ \text{green} \end{array} \begin{array}{c} \begin{array}{ccc} \text{apple juice} & \text{coffee} & \text{tea} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

(b) (0,1)-matrix

Three Relations

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

Operations with relations

- Intersection
- Union
- Difference
- Selection
- Projection
- Join
- Composition

Relations are Local Functions

- Relations are special case of a Local function

$$f : \prod_{x_i \in Y} D_i \rightarrow A$$

where

var(f) = $Y \subseteq X$: scope of function f

A: is a set of valuations

- In constraint networks: functions are boolean

x_1	x_2	f
a	a	true
a	b	false
b	a	false
b	b	true

relation →

x_1	x_2
a	a
b	b

Example of Set Operations: intersection, union, and difference applied to relations.

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

x_1	x_2	x_3
b	b	c
c	b	c

(a) $R \cap R'$

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s
c	n	n

(b) $R \cup R'$

x_1	x_2	x_3
a	b	c
c	b	s

(b) $R - R'$

Selection, Projection, and Join

x_1	x_2	x_3
a	b	c
b	b	c
c	b	c
c	b	s

(a) Relation R

x_1	x_2	x_3
b	b	c
c	b	c
c	n	n

(b) Relation R'

x_2	x_3	x_4
a	a	1
b	c	2
b	c	3

(c) Relation R''

x_1	x_2	x_3
b	b	c
c	b	c

(a) $\sigma_{x_3=c}(R')$

x_2	x_3
b	c
n	n

(b) $\pi_{\{x_2,x_3\}}(R')$

x_1	x_2	x_3	x_4
b	b	c	2
b	b	c	3
c	b	c	2
c	b	c	3

(c) $R' \bowtie R''$

Combination

- Join : $f \bowtie g$

x_1	x_2
a	a
b	b

$\bowtie$

x_2	x_3
a	a
a	b
b	a

$=$

x_1	x_2	x_3
a	a	a
a	a	b
b	b	a

- Logical AND: $f \wedge g$

x_1	x_2	f
a	a	true
a	b	false
b	a	false
b	b	true

$\wedge$

x_2	x_3	g
a	a	true
a	b	true
b	a	true
b	b	false

$=$

x_1	x_2	x_3	h
a	a	a	true
a	a	b	true
a	b	a	false
a	b	b	false
b	a	a	false
b	a	b	false
b	b	a	true
b	b	b	false

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- ✓ **Representing constraints/ Languages**
- ✓ **Constraint graphs**
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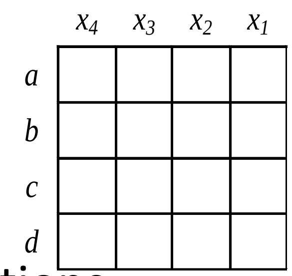
Modeling; Representing a problems

- If a CSP M = <X,D,C> represents a real problem P , then every solution of M corresponds to a solution of P and every solution of P can be derived from at least one solution of M



- The variables and values of M represent entities in P

- The constraints of M ensure the correspondence between solutions

- The aim is to find a model M that can be solved as quickly as possible

- **goal of modeling: choose a set of variables and values that allows the constraints to be expressed easily and concisely**

Given a proposition theory $\boldsymbol{\varphi = \{(A \vee B), (C \vee \neg B)\}}$ does it have a model?

Can it be encoded as a constraint network?

Variables: {A, B, C}

Domains: $D_A = D_B = D_C = \{0, 1\}$

Relations:

A	B
0	1
1	0
1	1

B	C
0	0
0	1
1	1

If this constraint network has a solution, then the propositional theory has a model

Constraint's representations

- Relation: allowed tuples

X	Y	Z
1	3	2
2	1	3

- Algebraic expression: $X + Y^2 \leq 10, X \neq Y$

- Propositional formula: $(a \vee b) \rightarrow \neg c$

- A decision tree, a procedure

- Semantics: by a relation

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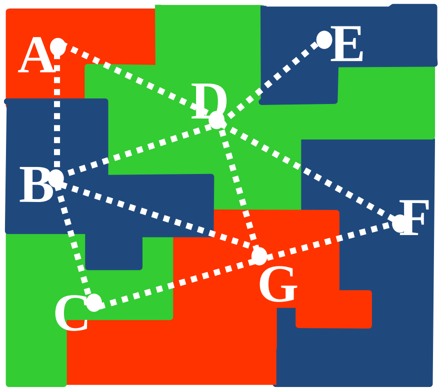
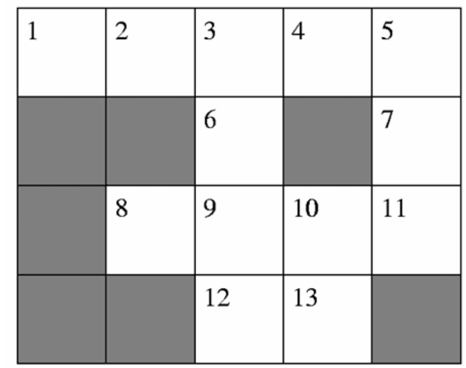
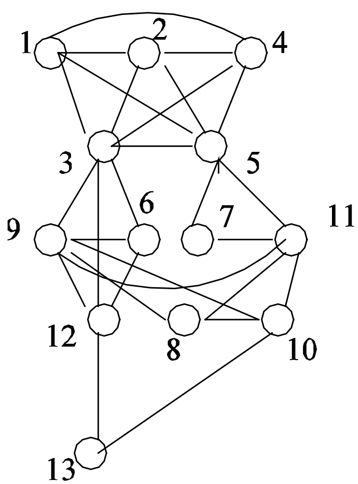


Constraint Graphs:

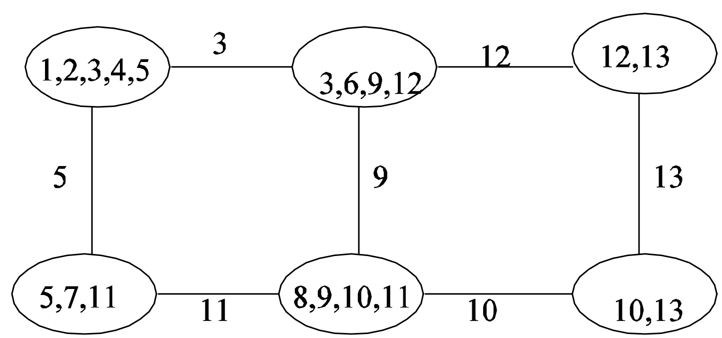
Primal, Dual and Hypergraphs



- A **(primal) constraint graph**: a node per variable, arcs connect constrained variables.
- A **dual constraint graph**: a node per constraint's scope, an arc connect nodes sharing variables =hypergraph



(a)



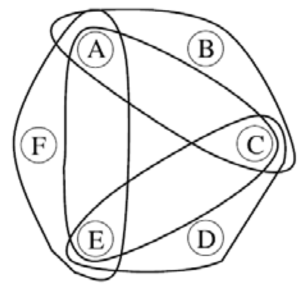
{HOSES, LASER, SHEET, SNAIL, STEER, ALSO, EARN, HIKE, IRON, SAME, EAT, LET, RUN, SUN, TEN, YES, BE, IT, NO, US}

(b)

Graph Concepts Reviews:

Hyper Graphs and Dual Graphs

A hypergraph



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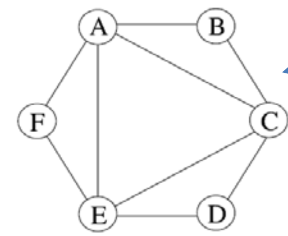

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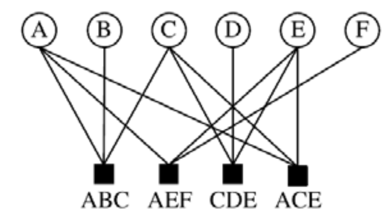
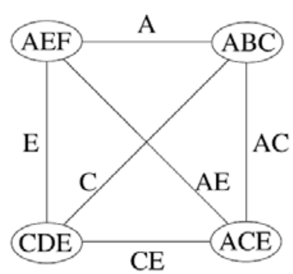
Primal graphs



(e)

Dual graph

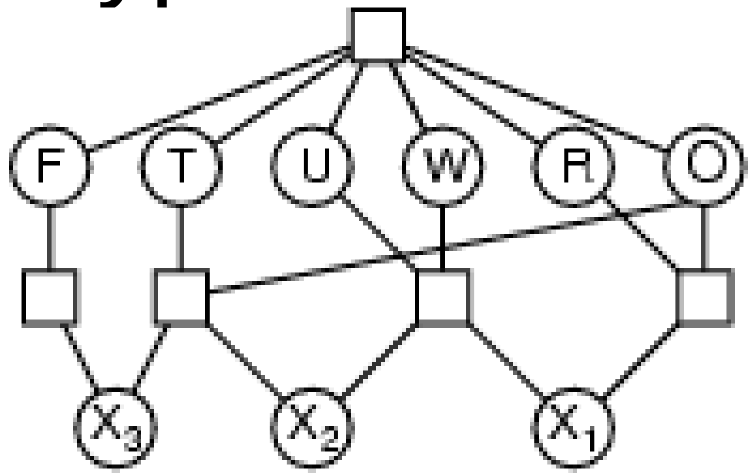


(c)

Factor graphs

Example: Cryptarithmetic

```
    T W O
  + T W O
  -------
  F O U R
```



Variables: $F\ T\ U\ W$
$R\ O\ X_1\ X_2\ X_3$

Domains: {$0,1,2,3,4,5,6,7,8,9$}

Constraints: *Alldiff (F,T,U,W,R,O)*

$O + O = R + 10 \cdot X_1$

$X_1 + W + W = U + 10 \cdot X_2$

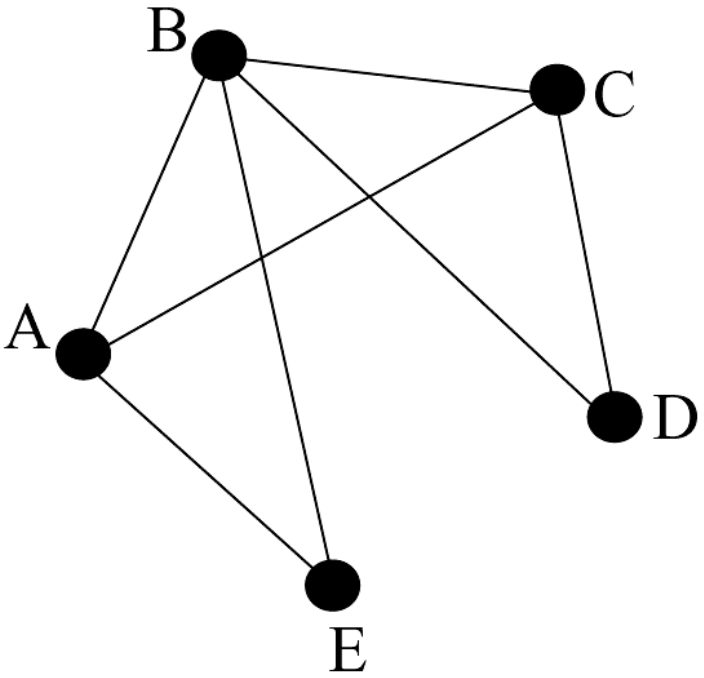
$X_2 + T + T = O + 10 \cdot X_3$

$X_3 = F,\ T \neq 0,\ F \neq 0$

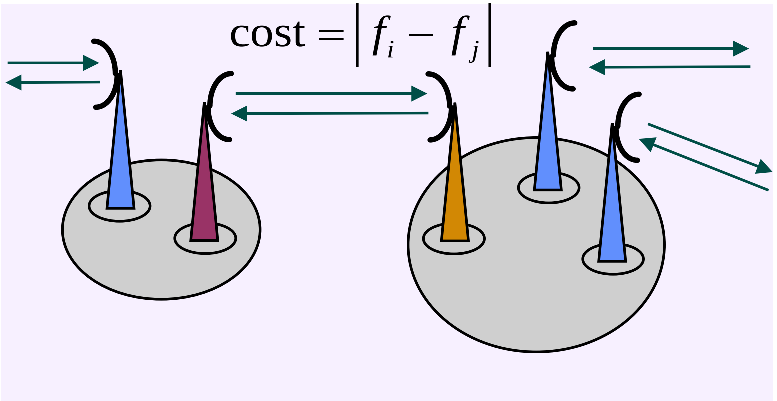
What is the primal graph?
What is the dual graph?

Propositional Satisfiability

$\varphi = \{(\neg C), (A \vee B \vee C), (\neg A \vee B \vee E), (\neg B \vee C \vee D)\}.$



Radio Link Assignment



Given a telecommunication network (where each communication link has various antennas) , assign a frequency to each antenna in such a way that all antennas may operate together without noticeable interference.

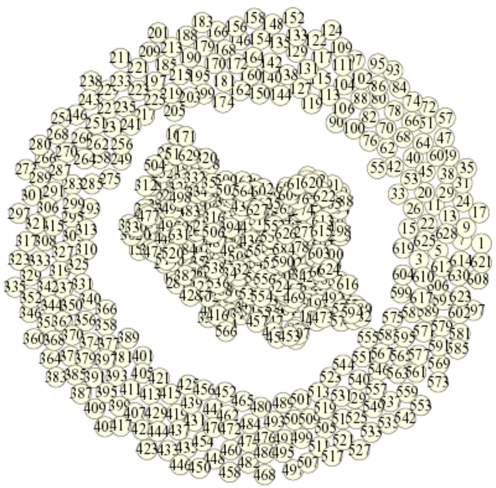
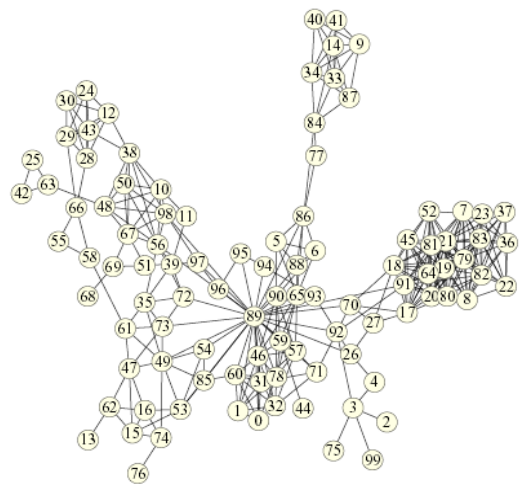
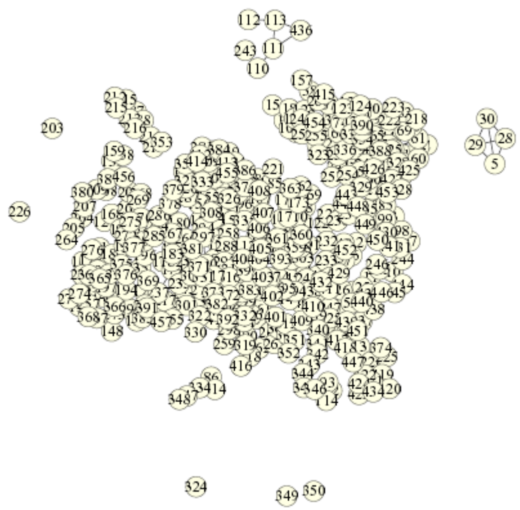
Encoding?

Variables: one for each antenna

Domains: the set of available frequencies

Constraints: the ones referring to the antennas in the same communication link

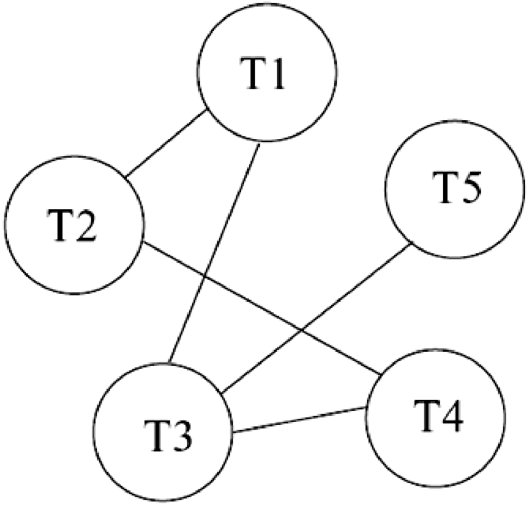
Constraint graphs of 3 instances of the Radio frequency assignment problem in CELAR’s benchmark

Scheduling problem

Five tasks: T1, T2, T3, T4, T5
Each one takes one hour to complete
The tasks may start at 1:00, 2:00 or 3:00
Requirements:

- T1 must start after T3
- T3 must start before T4 and after T5
- T2 cannot execute at the same time as T1 or T4
- T4 cannot start at 2:00



Encoding?

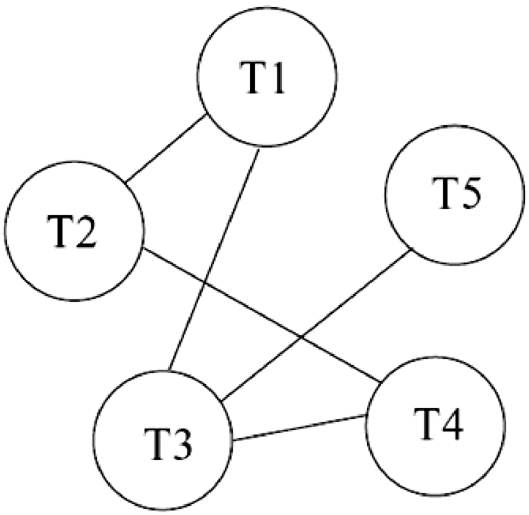
Variables: one for each task

Domains: $D_{T1} = D_{T2} = D_{T3} = D_{T3} = \{1{:}00, 2{:}00, 3{:}00\}$

Constraints:

T4
1:00
3:00

The constraint graph and relations of scheduling problem



Unary constraint

$D_{T4} = \{1{:}00,\ 3{:}00\}$

Binary constraints

$R_{\{T1,T2\}}$: $\{(1{:}00,2{:}00), (1{:}00,3{:}00), (2{:}00,1{:}00), (2{:}00,3{:}00), (3{:}00,1{:}00), (3{:}00,2{:}00)\}$

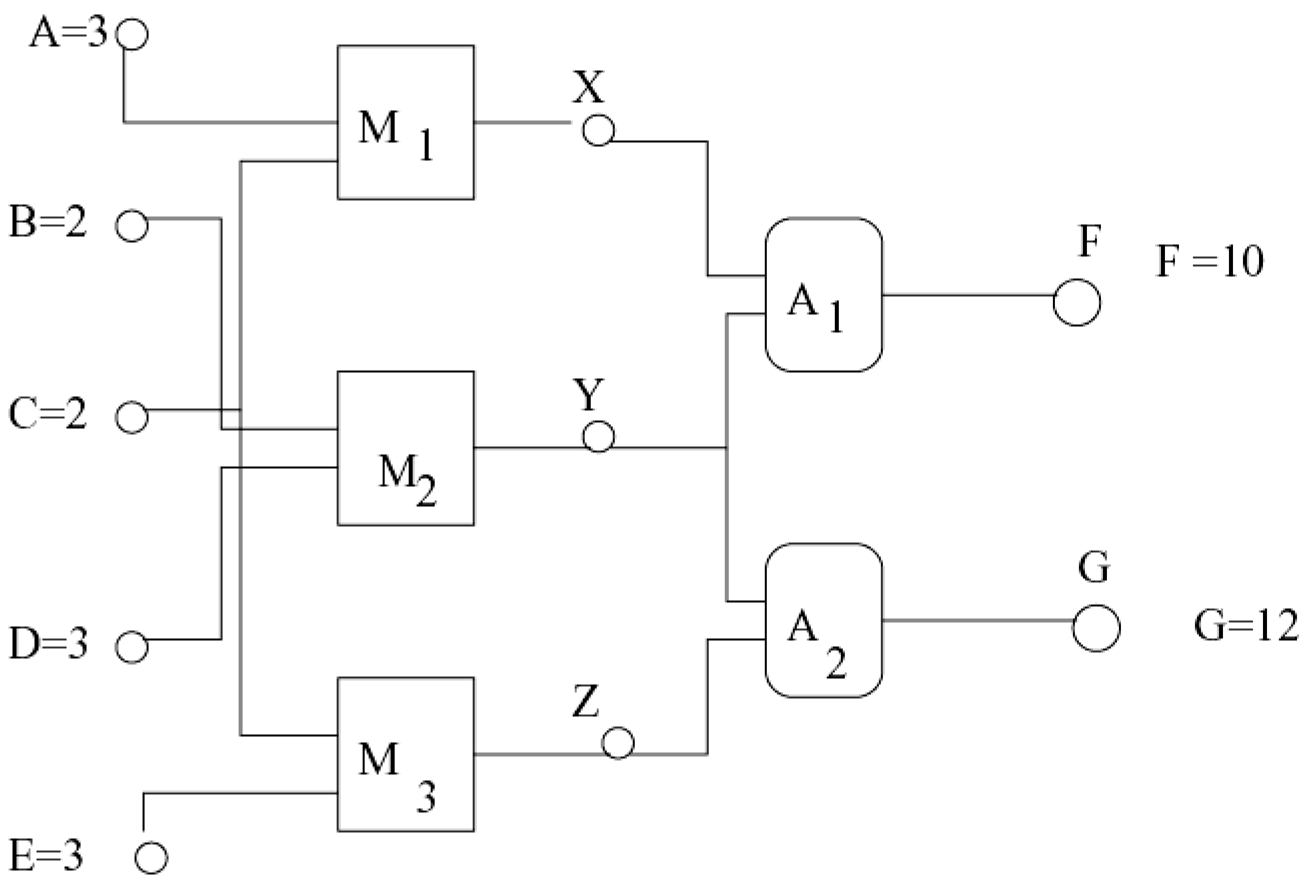
$R_{\{T1,T3\}}$: $\{(2{:}00,1{:}00), (3{:}00,1{:}00), (3{:}00,2{:}00)\}$

$R_{\{T2,T4\}}$: $\{(1{:}00,2{:}00), (1{:}00,3{:}00), (2{:}00,1{:}00), (2{:}00,3{:}00), (3{:}00,1{:}00), (3{:}00,2{:}00)\}$

$R_{\{T3,T4\}}$: $\{(1{:}00,2{:}00), (1{:}00,3{:}00), (2{:}00,3{:}00)\}$

$R_{\{T3,T5\}}$: $\{(2{:}00,1{:}00), (3{:}00,1{:}00), (3{:}00,2{:}00)\}$

A combinatorial circuit: *M* is a multiplier, *A* is an adder

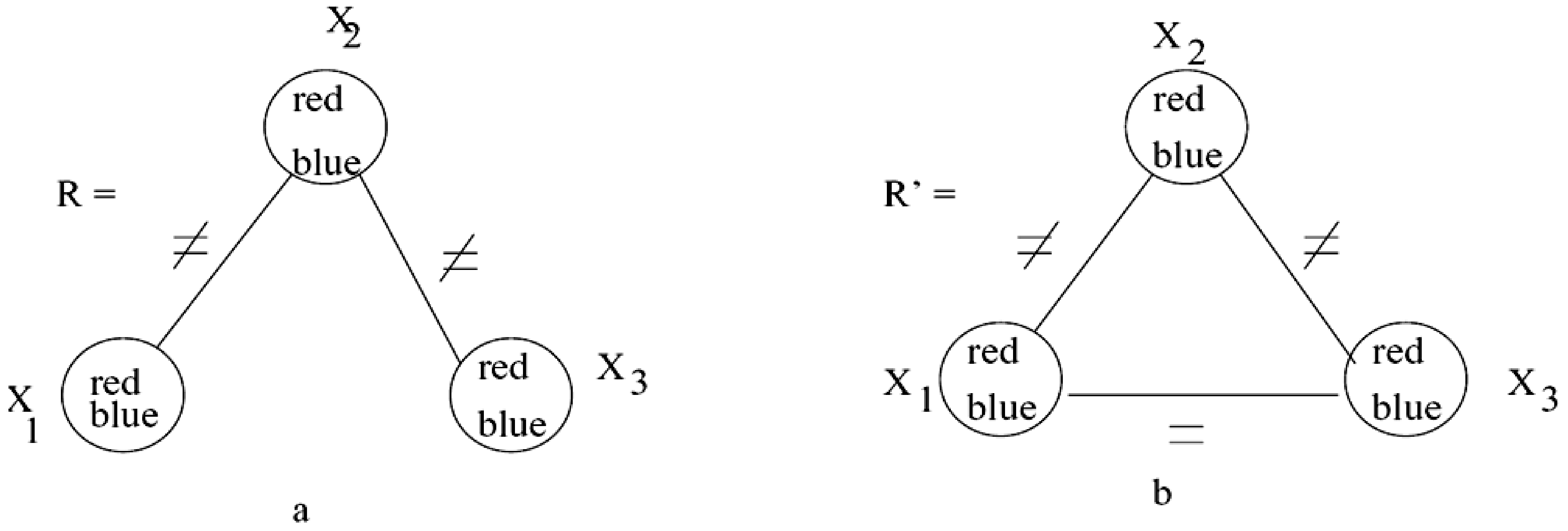


Outline

- ✓ Motivation, applications, history
- ✓ CSP: Definition, representation and simple modeling examples
- ✓ Mathematical concepts (relations, graphs)
- ✓ Representing constraints
- ✓ Constraint graphs
- ✓ **The binary Constraint Networks properties**

Properties of Binary Constraint Networks

A graph $\Re$ to be colored by two colors,
an equivalent representation $\Re$' having a newly inferred constraint between *x*1 and *x*3.



Equivalence and deduction with constraints (composition)

Composition of relations (*Montanari'74*)

Input: two binary relations $\boldsymbol{R}_{ab}$ and $\boldsymbol{R}_{bc}$ with 1 variable in common.

Output: a new induced relation $\boldsymbol{R}_{ac}$ (to be combined by intersection to a pre-existing relation between them, if any).

Bit-matrix operation: matrix multiplication

$$R_{ac} = R_{ab} \cdot R_{bc}$$

$$R_{ab} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R_{bc} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R_{ac} = ?$$

Equivalence, Redundancy, Composition

- Equivalence: Two constraint networks are equivalent if they have the same set of solutions.
- Composition in matrix notation

$$R_{xz} = R_{xy} \cdot R_{yz}$$

- Composition in relational operation

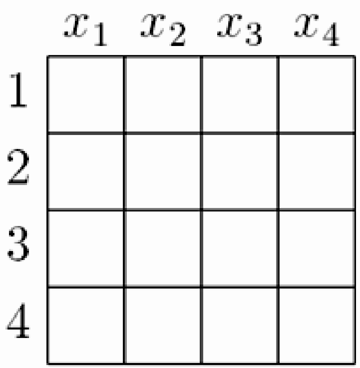
$$R_{xz} = \pi_{xz}(R_{xy} \bowtie R_{yz})$$

Relations vs Networks

- Can we represent by binary constraint networks the relations
- R(x1,x2,x3) = {(0,0,0)(0,1,1)(1,0,1)(1,1,0)}
- R(X1,x2,x3,x4) = {(1,0,0,0)(0,1,0,0) (0,0,1,0)(0,0,0,1)}
- Number of relations 2^{k^n}
- Number of networks: $2^{n^2 k}$
- Most relations cannot be represented by binary constraint networks

The N-queens constraint network
Is there a tighter network?

The network has four variables, all with domains *Di* = {1, 2, 3, 4}.
(a) The labeled chess board. (b) The constraints between variables.



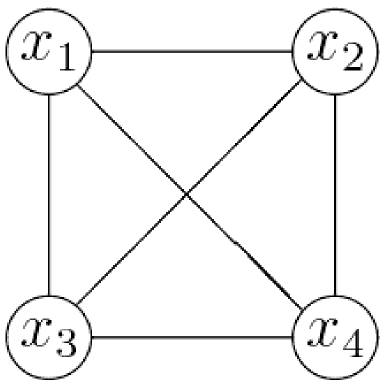
(a)

$$
\begin{aligned}
R_{12} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}\\
R_{13} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\}\\
R_{14} &= \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (3,4)\\
&\quad\ (4,2),\ (4,3)\}\\
R_{23} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}\\
R_{24} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\}\\
R_{34} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}
\end{aligned}
$$

(b)

Solutions are: (2,4,1,3) (3,1,4,2)

The 4-queens constraint network:
(a) The constraint graph. (b) The minimal binary constraints.
(c) The minimal unary constraints (the domains).



(a)

$$
\begin{aligned}
M_{12} &= \{(2,4),\ (3,1)\}\\
M_{13} &= \{(2,1),\ (3,4)\}\\
M_{14} &= \{(2,3),\ (3,2)\}\\
M_{23} &= \{(1,4),\ (4,1)\}\\
M_{24} &= \{(1,2),\ (4,3)\}\\
M_{34} &= \{(1,3),\ (4,2)\}
\end{aligned}
$$

(b)

$$
\begin{aligned}
D_1 &= \{\boxed{2}\,3\}\\
D_2 &= \{1,4\}\\
D_3 &= \{1,4\}\\
D_4 &= \{\boxed{2},3\}
\end{aligned}
$$

(c)

Solutions are: (2,4,1,3) (3,1,4,2)

The projection networks

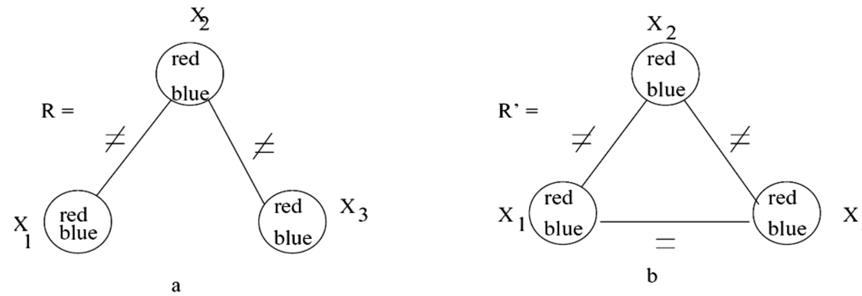
- The projection network of a relation is obtained by projecting it onto each pair of its variables (yielding a binary network).
- *Relation = {(1,1,2)(1,2,2)(1,2,1)}*
  - *What is the projection network?*
- What is the relationship between a relation and its projection network?
- {(1,1,2)(1,2,2)(2,1,3)(2,2,2)} are the solutions of its projection network?

Projection network (continued)

- **Theorem**: *Every relation is included in the set of solutions of its projection network.*
- **Theorem**: *The projection network is the tightest upper bound binary networks representation of the relation.*

Therefore, If a network cannot be represented by its projection network it has no binary network representation

Partial Order between networks, The Minimal Network



Definition 2.3.10 *Given two binary networks, $\mathcal{R}'$ and $\mathcal{R}$, on the same set of variables $x_1, ..., x_n$, $\mathcal{R}'$ is* at least as tight as $\mathcal{R}$ *iff for every i and j, $R'_{ij} \subseteq R_{ij}$.*

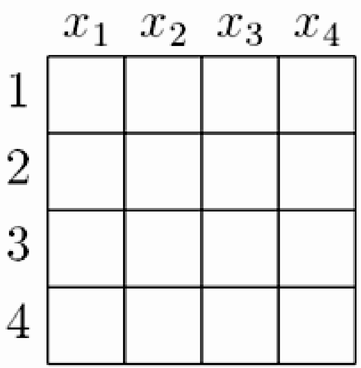
- An intersection of two networks is tighter (as tight) than both
- An intersection of two equivalent networks is equivalent to both

Definition 2.3.14 *Let $\{\mathcal{R}_1, ...\mathcal{R}_l\}$ be the set of all networks equivalent to $\mathcal{R}_0$ and let $\rho = sol(\mathcal{R}_0)$. Then the minimal network M of $\mathcal{R}_0$ is defined by $M(\mathcal{R}_0) = \cap_{i=1}^{l} \mathcal{R}_i$.*

Theorem 2.3.15 *For every binary network $\mathcal{R}$ s.t. $\rho = sol(\mathcal{R})$, $M(\rho) = P(\rho)$.*

The N-queens constraint network.

The network has four variables, all with domains *Di* = {1, 2, 3, 4}.
(a) The labeled chess board. (b) The constraints between variables.

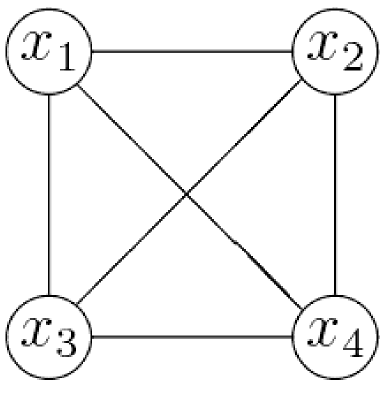


(a)

$$
\begin{aligned}
R_{12} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{13} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{14} &= \{(1,2),\ (1,3),\ (2,1),\ (2,3),\ (2,4),\ (3,1),\ (3,2),\ (3,4) \\
&\qquad (4,2),\ (4,3)\} \\
R_{23} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\} \\
R_{24} &= \{(1,2),\ (1,4),\ (2,1),\ (2,3),\ (3,2),\ (3,4),\ (4,1),\ (4,3)\} \\
R_{34} &= \{(1,3),\ (1,4),\ (2,4),\ (3,1),\ (4,1),\ (4,2)\}
\end{aligned}
$$

(b)

The 4-queens constraint network:
(a) The constraint graph. (b) The minimal binary constraints.
(c) The minimal unary constraints (the domains).



(a)

$$
\begin{aligned}
M_{12} &= \{(2,4),\ (3,1)\}\\
M_{13} &= \{(2,1),\ (3,4)\}\\
M_{14} &= \{(2,3),\ (3,2)\}\\
M_{23} &= \{(1,4),\ (4,1)\}\\
M_{24} &= \{(1,2),\ (4,3)\}\\
M_{34} &= \{(1,3),\ (4,2)\}
\end{aligned}
$$

(b)

$$
\begin{aligned}
D_1 &= \{\boxed{2},3\}\\
D_2 &= \{1,4\}\\
D_3 &= \{1,4\}\\
D_4 &= \{\boxed{2},3\}
\end{aligned}
$$

(c)

Solutions are: (2,4,1,3) (3,1,4,2)