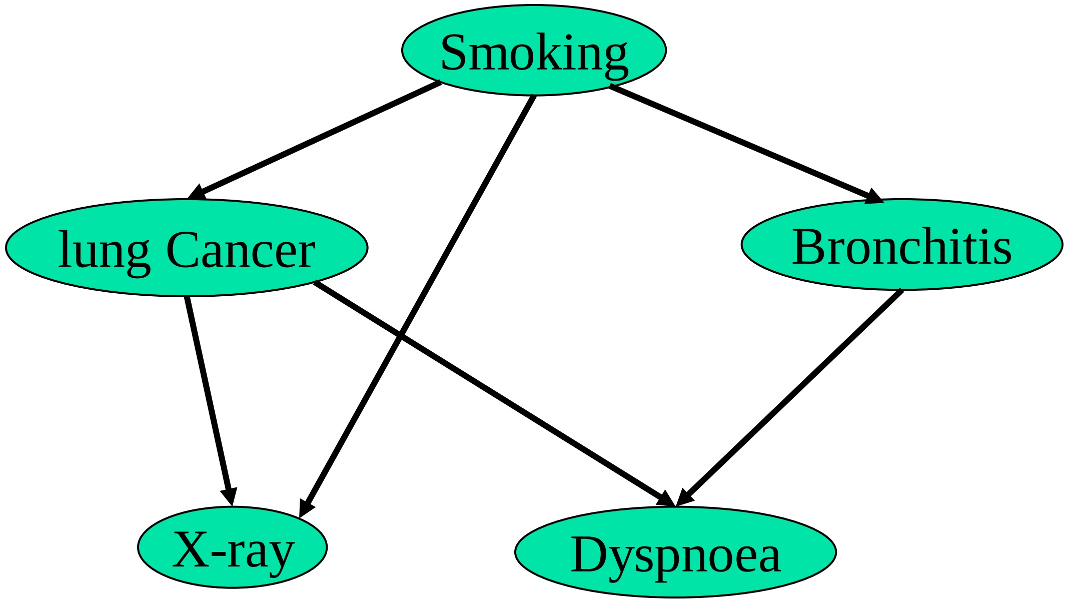
# Exact Inference Algorithms Bucket-elimination

COMPSCI 276, Fall 2014

Class 5: Rina Dechter

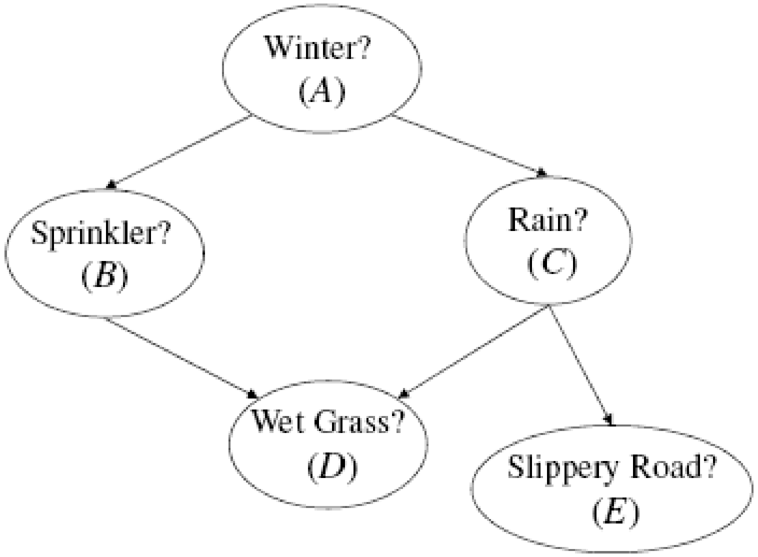
(Reading: Dechter chapters 4 , Darwiche chapter 6, Dechter Section 3.4)

# Belief Updating



P (lung cancer=yes | smoking=no, dyspnoea=yes ) = ?

## A Bayesian Network



| $A$ | $\Theta_A$ |
|---|---|
| true | .6 |
| false | .4 |

| $A$ | $B$ | $\Theta_{B\|A}$ |
|---|---|---|
| true | true | .2 |
| true | false | .8 |
| false | true | .75 |
| false | false | .25 |

| $A$ | $C$ | $\Theta_{C\|A}$ |
|---|---|---|
| true | true | .8 |
| true | false | .2 |
| false | true | .1 |
| false | false | .9 |

| $B$ | $C$ | $D$ | $\Theta_{D\|BC}$ |
|---|---|---|---|
| true | true | true | .95 |
| true | true | false | .05 |
| true | false | true | .9 |
| true | false | false | .1 |
| false | true | true | .8 |
| false | true | false | .2 |
| false | false | true | 0 |
| false | false | false | 1 |

| $C$ | $E$ | $\Theta_{E\|C}$ |
|---|---|---|
| true | true | .7 |
| true | false | .3 |
| false | true | 0 |
| false | false | 1 |

# Queries

1. **Posterior marginals, or belief updating.** For every $X_i$ not in $E$ the belief is defined by $bel(X_i) = P_{\mathcal{B}}(X_i|e)$.

$$P(X_i|e) = \sum_{\mathbf{X}-X_i} \prod_j P(X_j|X_{pa_j}, e)$$

2. The **probability of evidence** is $P_{\mathcal{B}}(E = e)$. Formally,

$$P_{\mathcal{B}}(E = e) = \sum_{\mathbf{X}} \prod_j P(X_j|X_{pa_j}, e)$$

3. The **most probable explanation** ($mpe$) is an assignment $\mathbf{x}^o = (x^o{}_1, ..., x^o{}_n)$ satisfying

$$\mathbf{x}^o = argmax_{\mathbf{X}} \mathbf{P}_{\mathcal{B}} = argmax_{\mathbf{X}} \prod_j P(X_j|X_{pa_j}, e).$$

The $mpe$ value is $\mathbf{P}_{\mathcal{B}}(\mathbf{x}^o)$, sometime also called $MAP$.

4. **Maximum a posteriori hypothesis ( marginal** $map$**).** Given a set of hypothesized variables $\mathbf{A} = \{A_1, ..., A_k\}$, $\mathbf{A} \subseteq \mathbf{X}$, the $map$ task is to find an assignment $\mathbf{a}^o = (a^o{}_1, ..., a^o{}_k)$ such that

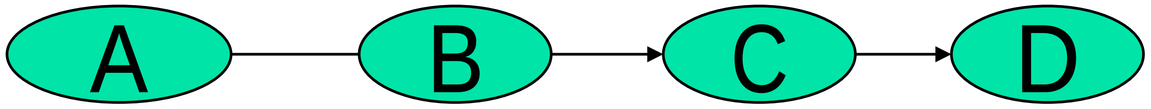
$$\mathbf{a}^o = argmax_{\mathbf{A}} \sum_{\mathbf{X}-\mathbf{A}} \mathbf{P}(\mathbf{X}|\mathbf{e}) = argmax_{\mathbf{A}} \sum_{\mathbf{X}-\mathbf{A}} \prod_j P(X_j|X_{pa_j}, e)$$

## Belief updating is NP-hard

- *Ea*ch sat formula can be mapped to a Bayesian network query.
- Example: (u,~v,w) and (~u,~w,y) sat?

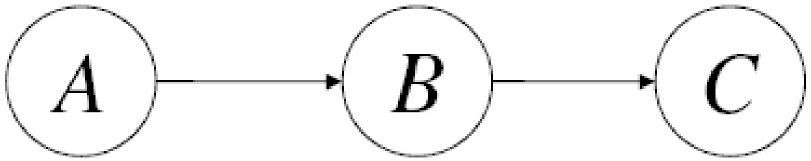
## A simple network

Given:



- How can we compute P(D)?,  P(D|A=0)? P(A|D=0)?
- Brute force O(k^4)
- Maybe O(4k^2)

## Elimination as a Basis for Inference


$$\begin{tikzcd} A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

| $A$ | $\Theta_A$ |
|---|---|
| true | .6 |
| false | .4 |

| $A$ | $B$ | $\Theta_{B\|A}$ |
|---|---|---|
| true | true | .9 |
| true | false | .1 |
| false | true | .2 |
| false | false | .8 |

| $B$ | $C$ | $\Theta_{C\|B}$ |
|---|---|---|
| true | true | .3 |
| true | false | .7 |
| false | true | .5 |
| false | false | .5 |

**To compute the prior marginal on variable $C$, $\Pr(C)$**

we first eliminate variable $A$ and then variable $B$

## Elimination as a Basis for Inference

- There are two factors that mention variable $A$, $\Theta_A$ and $\Theta_{B|A}$
- We multiply these factors first and then sum out variable $A$ from the resulting factor.
- Multiplying $\Theta_A$ and $\Theta_{B|A}$:

| $A$ | $B$ | $\Theta_A \Theta_{B\|A}$ |
|---|---|---|
| true | true | .54 |
| true | false | .06 |
| false | true | .08 |
| false | false | .32 |

- Summing out variable $A$:

| $B$ | $\sum_A \Theta_A \Theta_{B\|A}$ |
|---|---|
| true | $.62 = .54 + .08$ |
| false | $.38 = .06 + .32$ |

## Elimination as a Basis for Inference

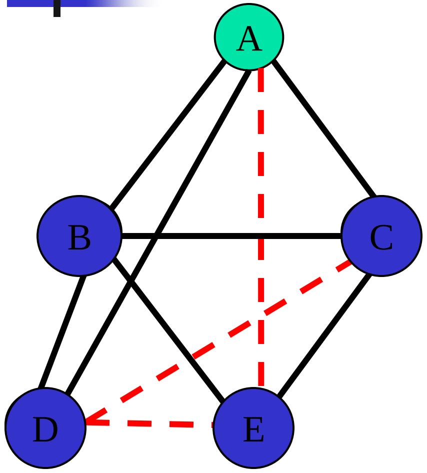
- We now have two factors, $\sum_A \Theta_A \Theta_{B|A}$ and $\Theta_{C|B}$, and we want to eliminate variable $B$
- Since $B$ appears in both factors, we must multiply them first and then sum out $B$ from the result.
- Multiplying:

| $B$ | $C$ | $\Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$ |
|---|---|---|
| true | true | .186 |
| true | false | .434 |
| false | true | .190 |
| false | false | .190 |

- Summing out:

| $C$ | $\sum_B \Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$ |
|---|---|
| true | .376 |
| false | .624 |

## Belief Updating: P(X|evidence)=?



"Moral" graph

$$P(a|e=0) \qquad P(a,e=0)=$$

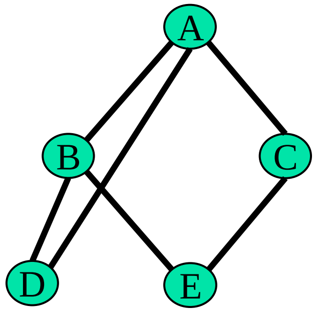
$$P(a)P(\underbrace{b|a)}P(c|a)\underbrace{P(d|b,a)P(e|b,c)}=$$

$$P(a) \qquad P(c|a) \qquad \underbrace{P(b|a)P(d|b,a)P(e|b,c)}$$

Variable Elimination

# Bucket Elimination

## Algorithm *BE-bel* (Dechter 1996)



$\longleftarrow$ Elimination operator

bucket B: $\overbrace{P(b|a) \quad P(d|b,a) \quad P(e|b,c)}$

bucket C: P(c|a) $\underbrace{\qquad\qquad}$

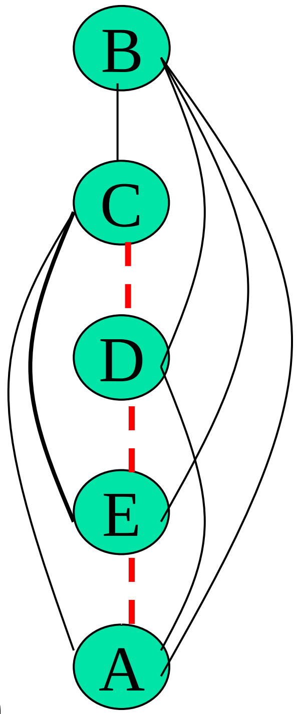
bucket D:

bucket E: e=0

bucket A: P(a)

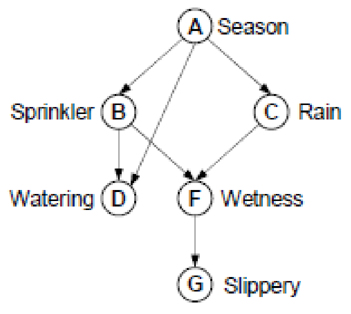
***P(a|e=0)***

W*=4
"induced width"
(max clique size)

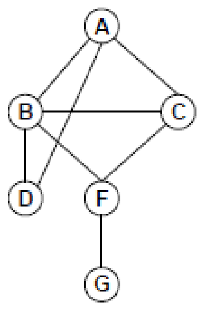


# A Bayesian Network

## Ordering: A,C,B,E,D,G



(a) Directed acyclic graph



(b) Moral graph

$$P(a, g=1) = \sum_{c,b,e,d,g=1} P(a,b,c,d,e,g) = \sum_{c,b,f,d,g=1} P(g|f)P(f|b,c)P(d|a,b)P(c|a)P(b|a)P(a).$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\sum_{f} P(f|b,c)\sum_{d} P(d|b,a)\sum_{g=1} P(g|f). \quad (4.1)$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\sum_{f} P(f|b,c)\lambda_G(f)\sum_{d} P(d|b,a). \quad (4.2)$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\lambda_D(a,b)\sum_{f} P(f|b,c)\lambda_G(f) \quad (4.3)$$

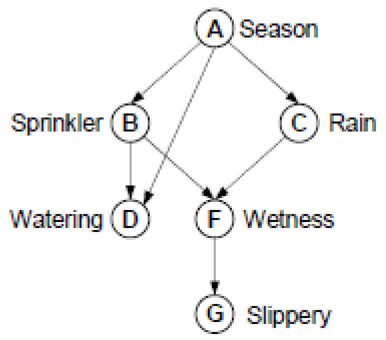
$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\lambda_D(a,b)\lambda_F(b,c) \quad (4.4)$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\lambda_B(a,c) \quad (4.5)$$

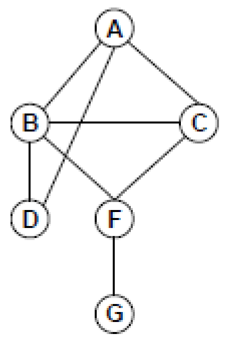
# A Bayesian Network

## ordering: A,C,B,F,D,G

$$\begin{tikzcd}
 & \sum\prod & \\
\text{Bucket G: } P(G|F) \arrow[ddr] & G=1 \arrow[dd] & \\
\text{Bucket D: } P(D|B,A) \arrow[ddr] & & \\
\text{Bucket F: } P(F|B,C) & \lambda_G(F) \arrow[dr] & \\
\text{Bucket B: } P(B|A) \arrow[dr] & \lambda_D(B,A) \arrow[d] & \lambda_F(B,C) \arrow[dl] \\
\text{Bucket C: } P(C|A) \arrow[dr] & \lambda_B(A,C) \arrow[d] & \\
\text{Bucket A: } P(A) \arrow[d] & \lambda_C(A) \arrow[dl] & \\
P(G=1) & &
\end{tikzcd}$$

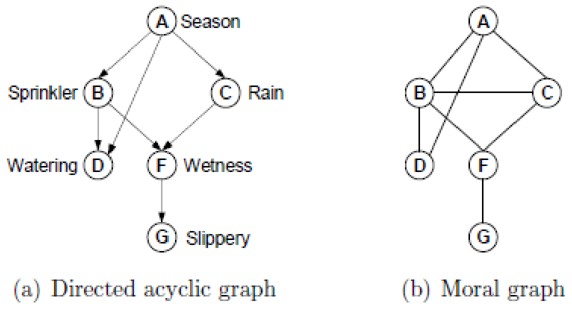


(a) Directed acyclic graph



(b) Moral graph

# A Different Ordering



$$P(a, g=1) = P(a)\sum_f \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b)P(f|b,c)\sum_{g=1} P(g|f)$$
$$= P(a)\sum_f \lambda_G(f)\sum_d \sum_c P(c|a)\sum_b P(b|a)\ P(d|a,b)P(f|b,c)$$
$$= P(a)\sum_f \lambda_G(f)\sum_d \sum_c P(c|a)\lambda_B(a,d,c,f)$$
$$= P(a)\sum_f \lambda_g(f)\sum_d \lambda_C(a,d,f)$$
$$= P(a)\sum_f \lambda_G(f)\lambda_D(a,f)$$
$$= P(a)\lambda_F(a)$$

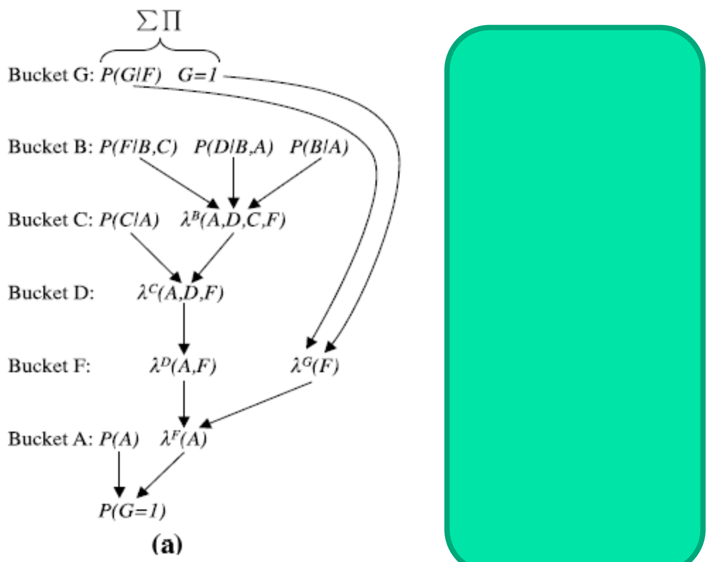
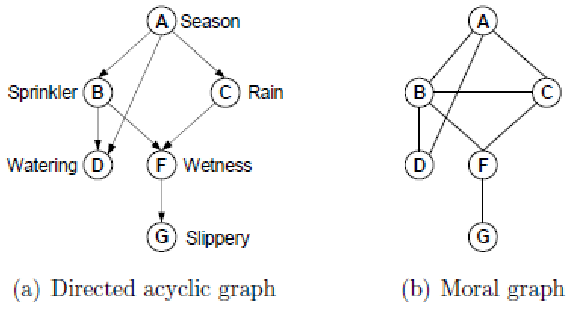
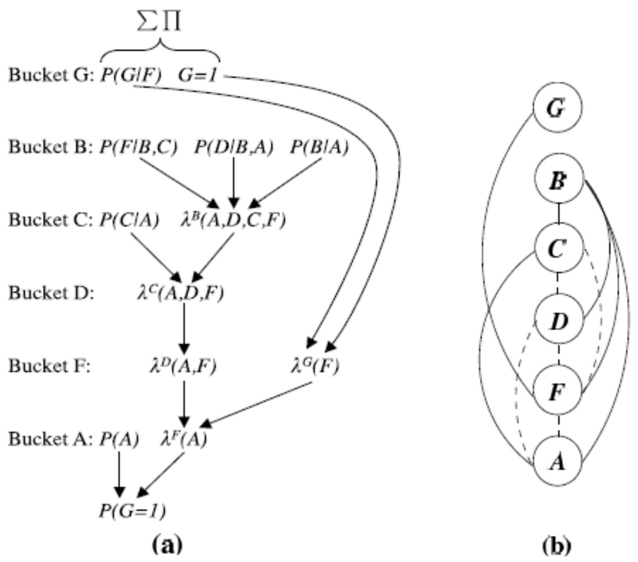

ΣΠ
Bucket G: P(G|F) G=1
Bucket B: P(F|B,C) P(D|B,A) P(B|A)
Bucket C: P(C|A) λ^B(A,D,C,F)
Bucket D: λ^C(A,D,F)
Bucket F: λ^D(A,F) λ^G(F)
Bucket A: P(A) λ^F(A)
P(G=1)
(a)

Figure 4.3: The bucket's output when processing along $d_2 = A, F, D, C, B, G$

# A Different Ordering



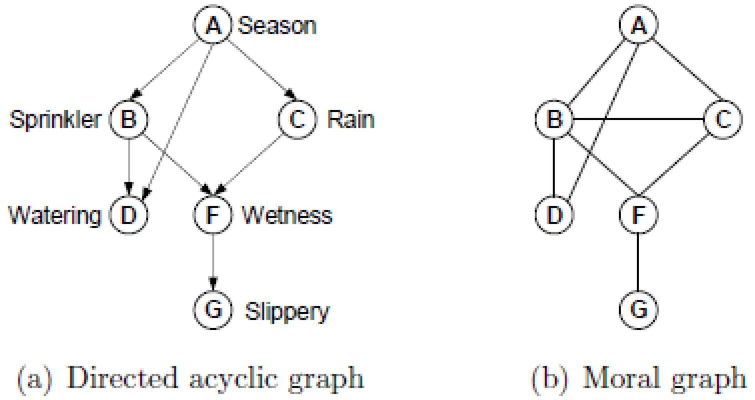
$$P(a, g=1) = P(a)\sum_f \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b)P(f|b,c) \sum_{g=1} P(g|f)$$
$$= P(a)\sum_f \lambda_G(f) \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b)P(f|b,c)$$
$$= P(a)\sum_f \lambda_G(f) \sum_d \sum_c P(c|a)\lambda_B(a,d,c,f)$$
$$= P(a)\sum_f \lambda_g(f) \sum_d \lambda_C(a,d,f)$$
$$= P(a)\sum_f \lambda_G(f)\lambda_D(a,f)$$
$$= P(a)\lambda_F(a)$$

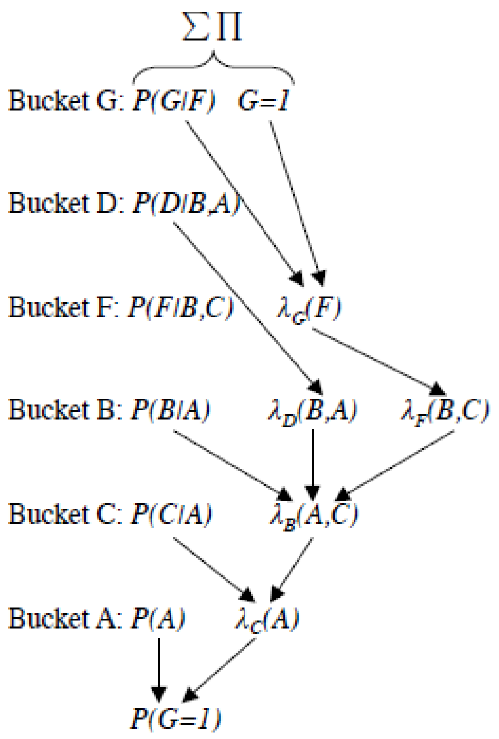


ΣΠ
Bucket G: P(G|F) G=1
Bucket B: P(F|B,C) P(D|B,A) P(B|A)
Bucket C: P(C|A) λ^B(A,D,C,F)
Bucket D: λ^C(A,D,F)
Bucket F: λ^D(A,F) λ^G(F)
Bucket A: P(A) λ^F(A)
P(G=1)
(a)
G
B
C
D
F
A
(b)

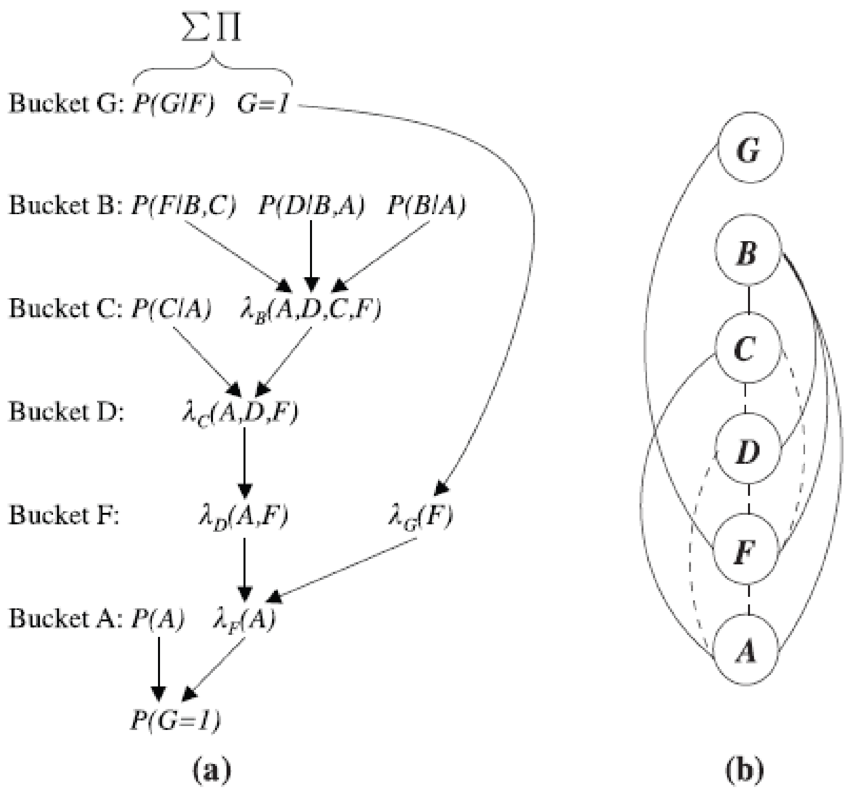
Figure 4.3: The bucket's output when processing along $d_2 = A, F, D, C, B, G$

# A Bayesian Network Processed Along 2 Orderings





ΣΠ
Bucket G: P(G|F) G=1
Bucket D: P(D|B,A)
Bucket F: P(F|B,C) λ_G(F)
Bucket B: P(B|A) λ_D(B,A) λ_F(B,C)
Bucket C: P(C|A) λ_B(A,C)
Bucket A: P(A) λ_C(A)
P(G=1)



ΣΠ
Bucket G: P(G|F) G=1
Bucket B: P(F|B,C) P(D|B,A) P(B|A)
Bucket C: P(C|A) λ_B(A,D,C,F)
Bucket D: λ_C(A,D,F)
Bucket F: λ_D(A,F) λ_G(F)
Bucket A: P(A) λ_F(A)
P(G=1)
(a)
G
B
C
D
F
A
(b)

Figure 4.4: The bucket's output when processing along $d_2 = A, F, D, C, B, G$.

## Factors: Sum-Out Operation

**The sum-out operation is commutative**

$$\sum_{Y}\sum_{X} f = \sum_{X}\sum_{Y} f$$

No need to specify the order in which variables are summed out.

**If a factor $f$ is defined over disjoint variables $\mathbf{X}$ and $\mathbf{Y}$**

then $\sum_{\mathbf{X}} f$ is said to marginalize variables $\mathbf{X}$

**If a factor $f$ is defined over disjoint variables $\mathbf{X}$ and $\mathbf{Y}$**

then $\sum_{\mathbf{X}} f$ is called the result of projecting $f$ on variables $\mathbf{Y}$

## Factors: Multiplication Operation

| $B$ | $C$ | $D$ | $f_1$ |
|---|---|---|---|
| true | true | true | .95 |
| true | true | false | .05 |
| true | false | true | .9 |
| true | false | false | .1 |
| false | true | true | .8 |
| false | true | false | .2 |
| false | false | true | 0 |
| false | false | false | 1 |

| $D$ | $E$ | $f_2$ |
|---|---|---|
| true | true | 0.448 |
| true | false | 0.192 |
| false | true | 0.112 |
| false | false | 0.248 |

The result of multiplying the above factors:

| $B$ | $C$ | $D$ | $E$ | $f_1(B,C,D)f_2(D,E)$ |
|---|---|---|---|---|
| true | true | true | true | $0.4256 = (.95)(.448)$ |
| true | true | true | false | $0.1824 = (.95)(.192)$ |
| true | true | false | true | $0.0056 = (.05)(.112)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| | | | | |
| false | false | false | false | $0.2480 = (1)(.248)$ |

## Factors: Multiplication Operation

**The result of multiplying factors $f_1(\mathbf{X})$ and $f_2(\mathbf{Y})$**

is another factor over variables $\mathbf{Z} = \mathbf{X} \cup \mathbf{Y}$:

$$(f_1 f_2)(\mathbf{z}) \overset{def}{=} f_1(\mathbf{x}) f_2(\mathbf{y}),$$

where $\mathbf{x}$ and $\mathbf{y}$ are compatible with $\mathbf{z}$; that is, $\mathbf{x} \sim \mathbf{z}$ and $\mathbf{y} \sim \mathbf{z}$

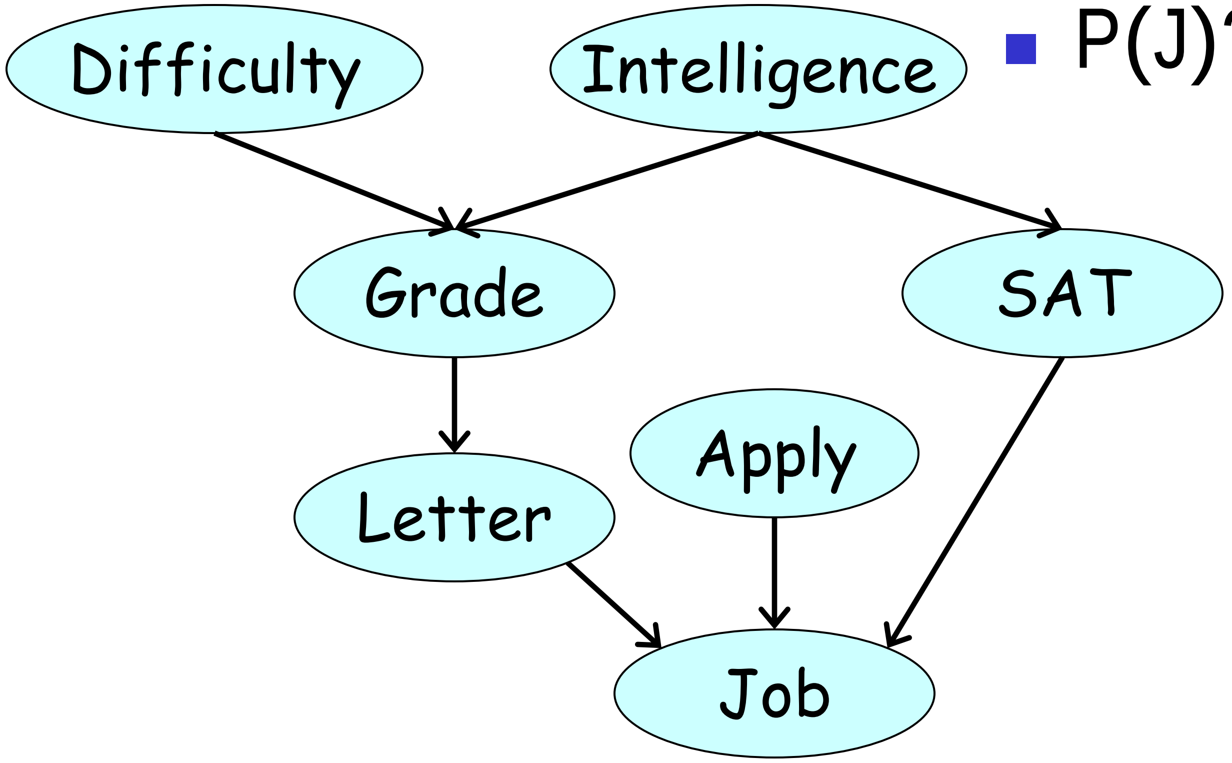
**Factor multiplication is commutative and associative**

It is meaningful to talk about multiplying a number of factors without specifying the order of this multiplication process.

```
ALGORITHM BE-BEL
Input: A belief network $\mathcal{B} = \langle \mathbf{X}, \mathbf{D}, \mathbf{P}_G, \prod \rangle$, an ordering $d = (X_1, \ldots, X_n)$ ; evidence $e$
output: The belief $P(X_1|e)$ and probability of evidence $P(e)$
1.  Partition the input functions (CPTs) into $bucket_1$, ..., $bucket_n$ as follows:
    for $i \leftarrow n$ downto 1, put in $bucket_i$ all unplaced functions mentioning $X_i$.
    Put each observed variable in its bucket. Denote by $\psi_i$ the product of input
    functions in $bucket_i$.
2.  backward: for $p \leftarrow n$ downto 1 do
3.  for all the functions $\psi_{S_0}, \lambda_{S_1}, \ldots, \lambda_{S_j}$ in $bucket_p$ do
        If (observed variable) $X_p = x_p$ appears in $bucket_p$,
        assign $X_p = x_p$ to each function in $bucket_p$ and then
        put each resulting function in the bucket of the *closest* variable in its scope.
        else,
4.          $\lambda_p \leftarrow \sum_{X_p} \psi_p \cdot \prod_{i=1}^{j} \lambda_{S_i}$
5.          place $\lambda_p$ in bucket of the latest variable in scope($\lambda_p$),
6.  return (as a result of processing $bucket_1$):
        $P(e) = \alpha = \sum_{X_1} \psi_1 \cdot \prod_{\lambda \in bucket_1} \lambda$
        $P(X_1|e) = \frac{1}{\alpha} \psi_1 \cdot \prod_{\lambda \in bucket_1} \lambda$
```

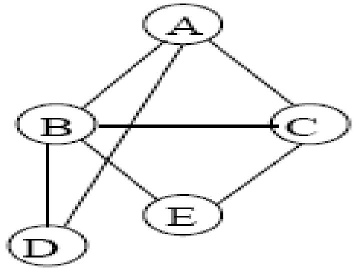
Figure 4.5: BE-bel: a sum-product bucket-elimination algorithm.

## Student Network example



- P(J)?

# Bucket Elimination and Induced Width



**Ordering: a, b, c, d, e**

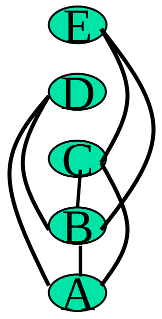
$bucket(E) = \quad P(e|b,c),\ \ e = 0$

$bucket(D) = \quad P(d|a,b)$

$bucket(C) = \quad P(c|a) \ \ || \ P(e=0|b,c)$

$bucket(B) = \quad P(b|a) \ \ || \ \ \lambda_D(a,b), \lambda_C(b,c)$

$bucket(A) = \quad P(a) \ \ || \ \ \lambda_B(a)$



**Ordering: a, e, d, c, b**

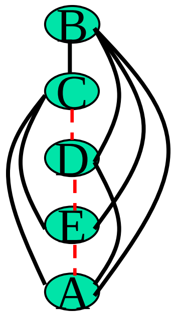
$bucket(B) = \quad P(e|b,c), P(d|a,b), P(b|a)$

$bucket(C) = \quad P(c|a) \ \ || \ \ \lambda_B(a,c,d,e)$

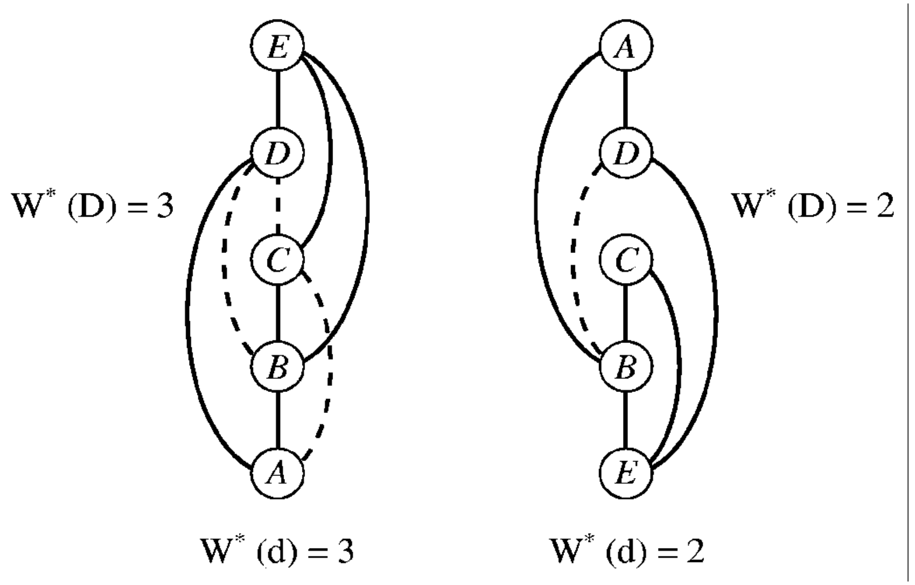
$bucket(D) = \quad || \ \lambda_C(a,d,e)$

$bucket(E) = \quad e = 0 \ \ || \ \lambda_D(a,c)$

$bucket(A) = \quad P(a) \ \ || \ \ \lambda_E(a)$



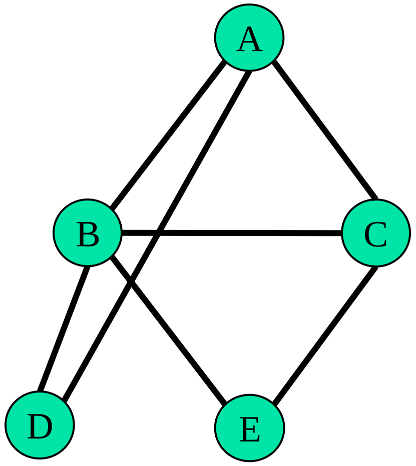
# The Induced-Width



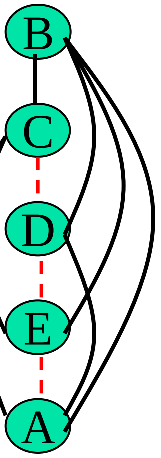
- width: is the max number of parents in the ordered graph
- Induced-width: width of induced graph: recursively connecting parents going from last node to first.
- Induced-width w*(d) = the max induced-width over all nodes
- Induced-width of a graph: max w*(d) over all d

## Complexity of elimination

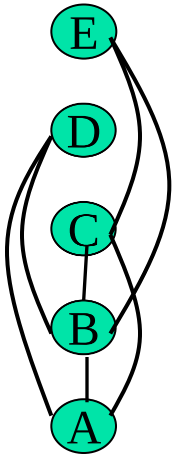
The effect of the ordering:



"Moral" graph

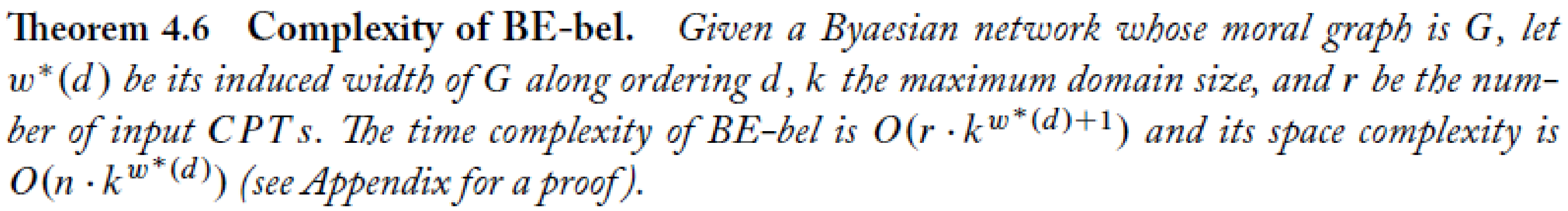


$$w^*(d_1) = 4$$



$$w^*(d_2) = 2$$

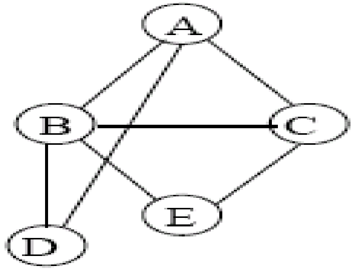
# Complexity of BE-bel



**Theorem 4.6 Complexity of BE-bel.** *Given a Byaesian network whose moral graph is G, let $w^*(d)$ be its induced width of G along ordering d, k the maximum domain size, and r be the number of input CPTs. The time complexity of BE-bel is $O(r \cdot k^{w^*(d)+1})$ and its space complexity is $O(n \cdot k^{w^*(d)})$ (see Appendix for a proof).*

More accurately: O(r exp(w*(d)) where r is the number of cpts.
For Bayesian networks r=n. For Markov networks?

## Handling Observations



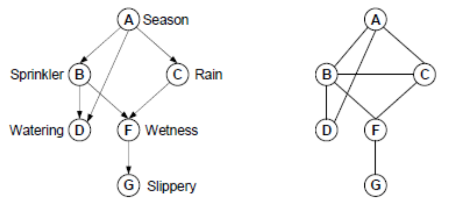
**Observing** $b = 1$

**Ordering: a, e, d, c, b**

$bucket(B) =$ $P(e|b,c), P(d|a,b), P(b|a), b = 1$
$bucket(C) =$ $P(c|a), \;||\; P(e|b=1,c)$
$bucket(D) =$ $||\; P(d|a,b=1)$
$bucket(E) =$ $e = 0 \;||\; \lambda_C(e,a)$
$bucket(A) =$ $P(a), \;||\; P(b=1|a)\; \lambda_D(a), \lambda_E(e,a)$

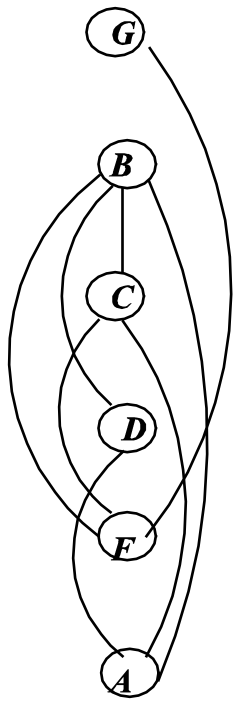
**Ordering: a, b, c, d, e**

$bucket(E) =$ $P(e|b,c),\; e = 0$
$bucket(D) =$ $P(d|a,b)$
$bucket(C) =$ $P(c|a) \;||\; \lambda_E(b,c)$
$bucket(B) =$ $P(b|a), b = 1 \;||\; \lambda_D(a,b), \lambda_C(a,b)$
$bucket(A) =$ $P(a) \;||\; \lambda_B(a)$

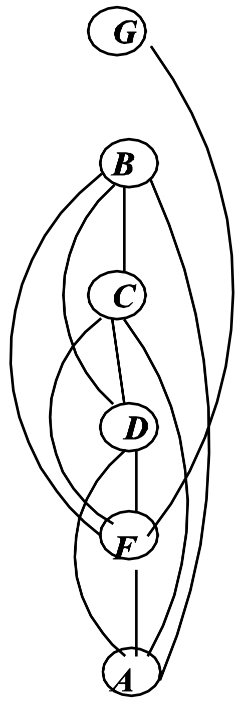


(a) Directed acyclic graph (b) Moral graph

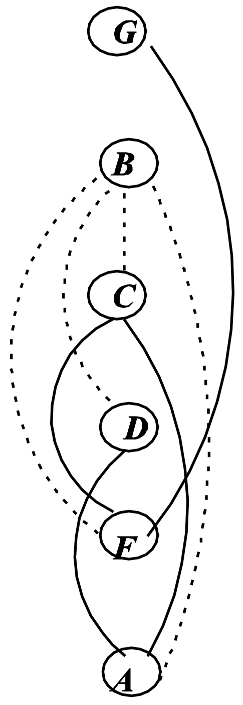
# The impact of observations



(a)



(b)



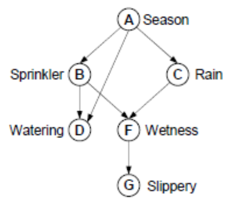
(c)

Ordered graph

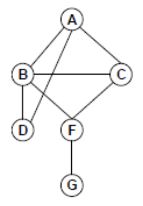
Induced graph

Ordered conditioned graph

## The impact of observations



(a) Directed acyclic graph



(b) Moral graph

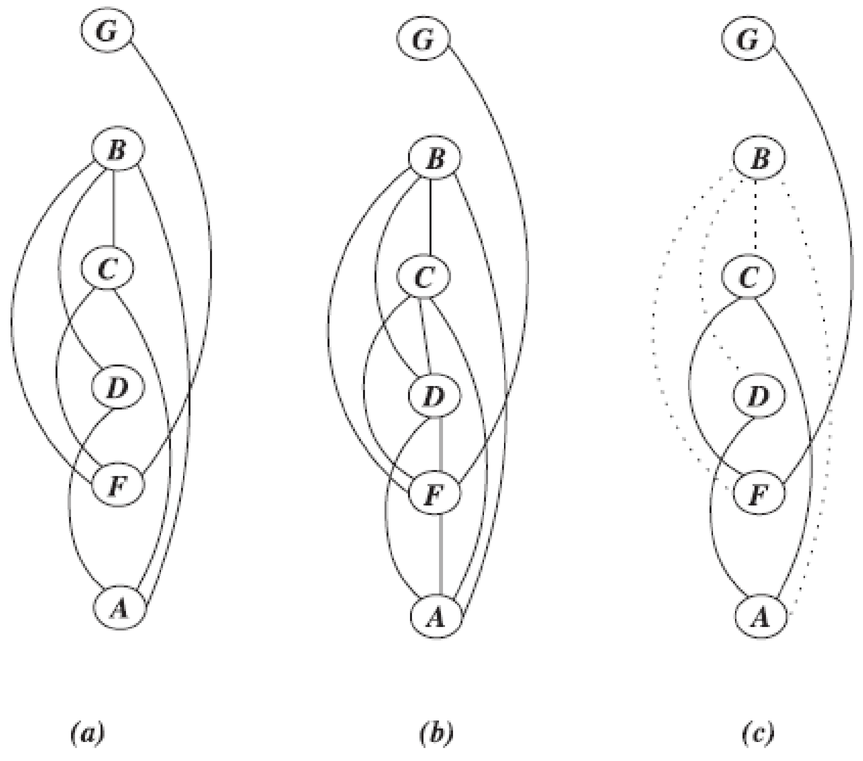
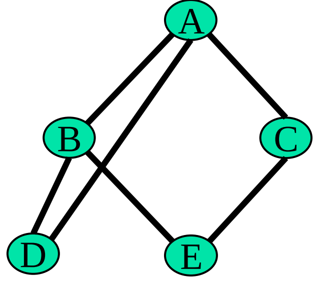


Figure 4.9: Adjusted induced graph relative to observing $B$.

Ordered graph Induced graph Ordered conditioned graph



"Moral" graph

## Irrelevant buckets for BE-BEL

Buckets that sum to 1 are **irrelevant**.
**Identification:** no evidence, no new functions.

**Recursive recognition :** ( $bel(a|e)$)

$bucket(E) =$ $P(e|b,c)$, $e = 0$
$bucket(D) =$ $P(d|a,b)$,...skipable bucket
$bucket(C) =$ $P(c|a)$
$bucket(B) =$ $P(b|a)$
$bucket(A) =$ $P(a)$

**Complexity:** Use induced width in moral graph without irrelevant nodes, then update for evidence arcs.

Use the ancestral graph only

## Pruning Nodes

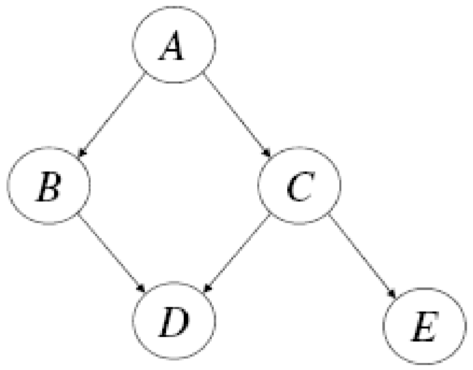
**Given a Bayesian network $\mathcal{N}$ and query $(\mathbf{Q}, \mathbf{e})$**

one can remove any leaf node (with its CPT) from the network as long as it does not belong to variables $\mathbf{Q} \cup \mathbf{E}$, yet not affect the ability of the network to answer the query correctly.

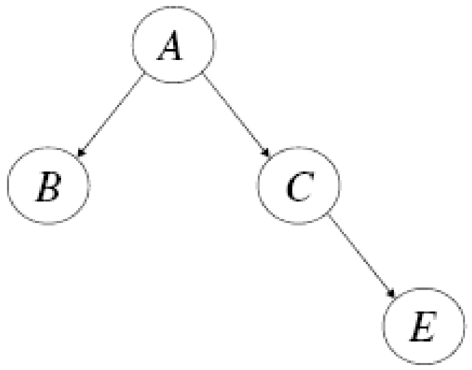
**If $\mathcal{N}' = \mathrm{pruneNodes}(\mathcal{N}, \mathbf{Q} \cup \mathbf{E})$**

then $\Pr(\mathbf{Q}, \mathbf{e}) = \Pr'(\mathbf{Q}, \mathbf{e})$, where $\Pr$ and $\Pr'$ are the probability distributions induced by networks $\mathcal{N}$ and $\mathcal{N}'$, respectively.

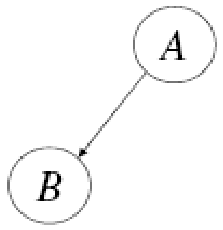
## Pruning Nodes: Example



network structure



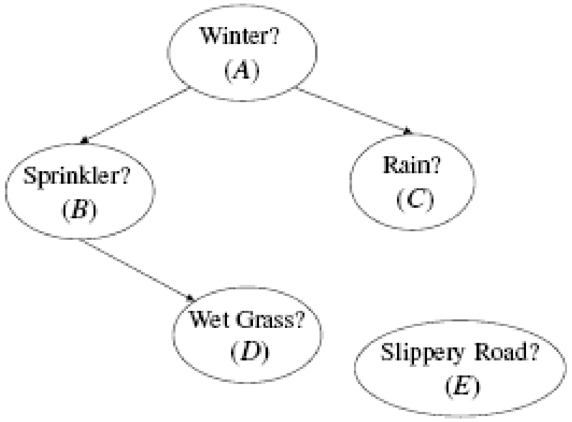
joint on $B, E$



joint on $B$

## Pruning Edges: Example

| $A$ | $B$ | $\Theta_{B\|A}$ |
|---|---|---|
| true | true | .2 |
| true | false | .8 |
| false | true | .75 |
| false | false | .25 |



| $A$ | $C$ | $\Theta_{C\|A}$ |
|---|---|---|
| true | true | .8 |
| true | false | .2 |
| false | true | .1 |
| false | false | .9 |

| $A$ | $\Theta_A$ |
|---|---|
| true | .6 |
| false | .4 |

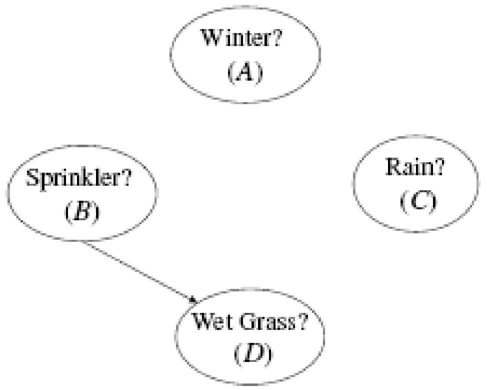
| $B$ | $D$ | $\sum_C \Theta_{D\|BC}^{C=\text{false}}$ |
|---|---|---|
| true | true | .9 |
| true | false | .1 |
| false | true | 0 |
| false | false | 1 |

| $E$ | $\sum_C \Theta_{E\|C}^{C=\text{false}}$ |
|---|---|
| true | 0 |
| false | 1 |

Evidence **e** : $C = \text{false}$

## Pruning Nodes and Edges: Example

| $B$ | $\Theta'_B = \sum_A \Theta_{B\|A}^{A=\text{true}}$ |
|---|---|
| true | .2 |
| false | .8 |



| $C$ | $\Theta'_C = \sum_A \Theta_{C\|A}^{A=\text{true}}$ |
|---|---|
| true | .8 |
| false | .2 |

| $A$ | $\Theta_A$ |
|---|---|
| true | .6 |
| false | .4 |

| $B$ | $D$ | $\Theta'_{D\|B} = \sum_C \Theta_{D\|BC}^{C=\text{false}}$ |
|---|---|---|
| true | true | .9 |
| true | false | .1 |
| false | true | 0 |
| false | false | 1 |

Query $\mathbf{Q} = \{D\}$ and $\mathbf{e} : A = \text{true}, C = \text{false}$

## A Mini-school in Lifted Algorithms for Probabilistic Programming at UCI

In the week of November 3^rd we will have a mini-school in lifted algorithms for probabilistic programming.

Location 4011/3-11 DBH. Afternoon discussions TBD.

Two Expert in this area will visit us:

Rodrigo de Salvo Braz: http://www.ai.sri.com/~braz/

Vibhav Gogate: http://www.hlt.utdallas.edu/~vgogate/

Schedule:

November 3: Rodrigo gives a talk 1-2 in AI/ML seminar. Afternoon: discussion
November 4: Rodrigo: 10-1, 3011: tutorial, afternoon: discussion
November 5: Vibhav: 10-12, 4011: tutorial, afternoon: discussion
November 6: Vibhav, 10-12, 4011: Tutorial, afternoon: discussion.

# Probabilistic Inference Tasks

- Belief updating:

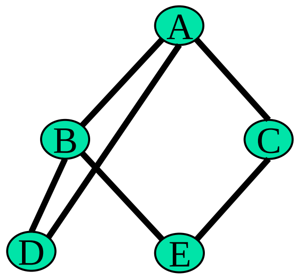
- Finding most probable explanation (MPE)

- Finding maximum a-posteriory hypothesis

$A \subseteq X$ :
**hypothesis variables**

## Finding

### Algorithm *BE-mpe*

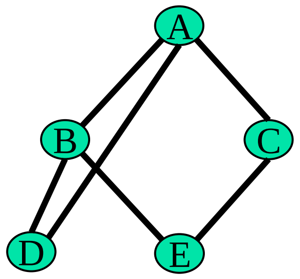


$\sum$ is replaced by ***max*** :

$$MPE = \max_{a,e,d,c,b} P(a)P(c \mid a)P(b \mid a)P(d \mid a,b)P(e \mid b,c)$$

# Finding

## Algorithm *elim-mpe* (Dechter 1996)



$\sum$ is replaced by ***max*** :

$$MPE = \max_{a,e,d,c,b} P(a)P(c|a)P(b|a)P(d|a,b)P(e|b,c)$$

Elimination operator

bucket B: P(b|a) P(d|b,a) P(e|b,c)

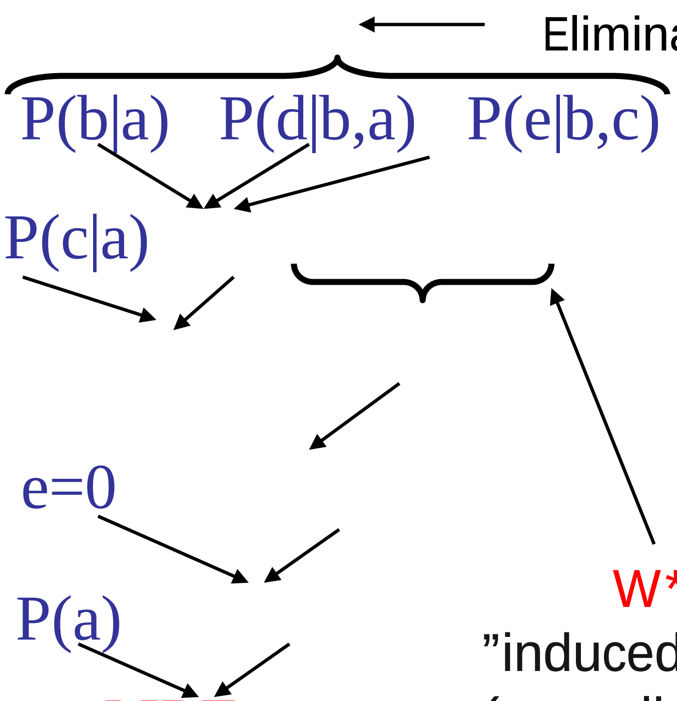
bucket C: P(c|a)

bucket D:

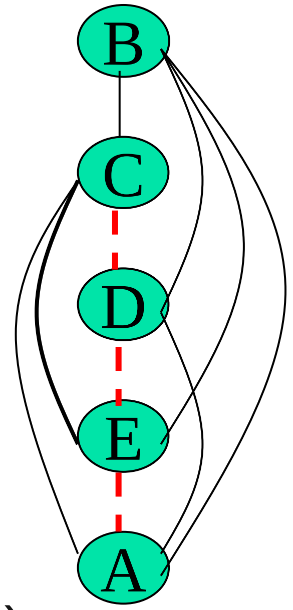
bucket E: e=0

bucket A: P(a)

***MPE***

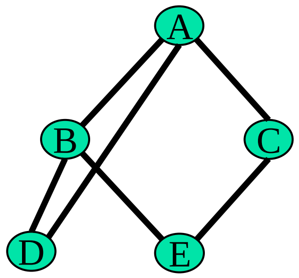


W*=4
"induced width"
(max clique size)



# Finding

## Algorithm *elim-mpe* (Dechter 1996)



$\sum$ is replaced by ***max*** **:**

$$MPE = \max_{a,e,d,c,b} P(a)P(c\,|\,a)P(b\,|\,a)P(d\,|\,a,b)P(e\,|\,b,c)$$

Elimination operator

bucket B: P(b|a) P(d|b,a) P(e|b,c)

bucket C: P(c|a)

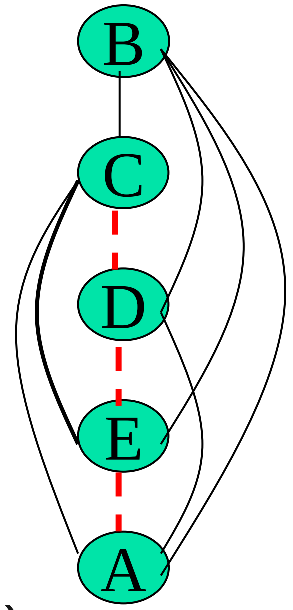
bucket D:

bucket E: e=0

bucket A: P(a)

***MPE***

W*=4
"induced width"
(max clique size)



# Generating the MPE-tuple

5. $b' = \arg\max_{b} P(b \mid a') \times P(d' \mid b, a') \times P(e' \mid b, c')$

4. $c' = \arg\max_{c} P(c \mid a') \times h^{B}(a', d', c, e')$

3. $d' = \arg\max_{d} h^{C}(a', d, e')$

2. $e' = 0$

1. $a' = \arg\max_{a} P(a) \cdot h^{E}(a)$

| | | | |
|---|---|---|---|
| B: | P(b\|a) | P(d\|b,a) | P(e\|b,c) |
| C: | P(c\|a) | $h^{B}(a,d,c,e)$ | |
| D: | | $h^{C}(a,d,e)$ | |
| E: | e=0 | $h^{D}(a,e)$ | |
| A: | P(a) | $h^{E}(a)$ | |

***Return*** $(a', b', c', d', e')$

**Algorithm BE-mpe**

**Input:** A belief network $\mathcal{B} =< X, D, G, \mathcal{P} >$, where $\mathcal{P} = \{P_1, ..., P_n\}$; an ordering of the variables, $d = X_1, ..., X_n$; observations $e$.

**Output:** The most probable assignment given the evidence.

1. **Initialize:** Generate an ordered partition of the conditional probability function, $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all functions whose highest variable is $X_i$. Put each observed variable in its bucket. Let $\psi_i$ be the input function in a bucket and let $h_i$ be the messages in the bucket.

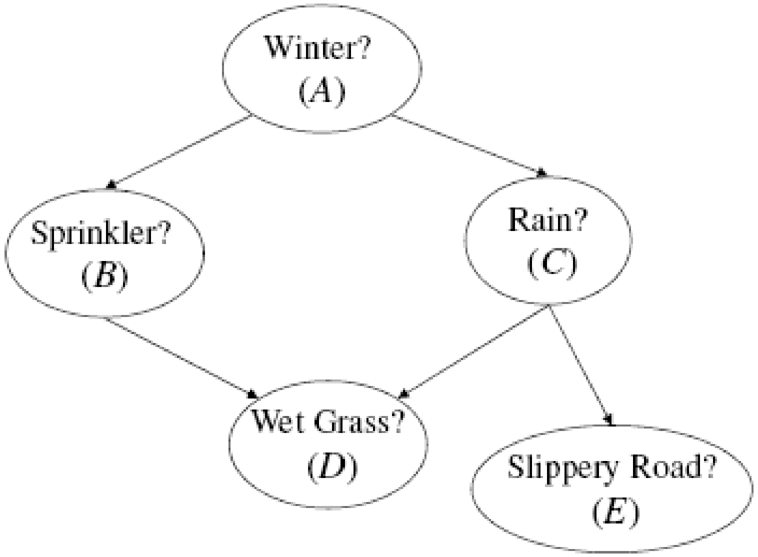
2. **Backward:** For $p \leftarrow n$ downto 1, do
for all the functions $h_1, h_2, ..., h_j$ in $bucket_p$, do

   - **If** (observed variable) $bucket_p$ contains $X_p = x_p$, assign $X_p = x_p$ to each function and put each in appropriate bucket.
   - **else,** $S_p \leftarrow \bigcup_{i=1}^{j} scope(h_i) \cup scope(\psi_p) - \{X_p\}$. Generate functions $h_p \Leftarrow \max_{X_p} \psi_p \cdot \Pi_{i=1}^{j} h_i$ Add $h_p$ to the bucket of the largest-index variable in $S_p$.

3. **Forward:**

   - Generate the mpe cost by maximizing over $X_1$, the product in $bucket_1$.
   - (generate an mpe tuple)
     For $i = 1$ to $n$ along $d$ do: Given $\overline{x}_{i-1} = (x_1, ..., x_{i-1})$ Choose $x_i = argmax_{X_i} \psi_i \cdot \Pi_{\{h_j \in\ bucket_i\}} h_j(\overline{x}_{i-1})$

## Try to compute MPE when E=0



| $A$ | $\Theta_A$ |
|---|---|
| true | .6 |
| false | .4 |

| $A$ | $B$ | $\Theta_{B\|A}$ |
|---|---|---|
| true | true | .2 |
| true | false | .8 |
| false | true | .75 |
| false | false | .25 |

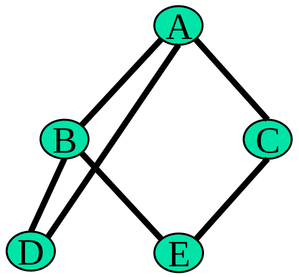
| $A$ | $C$ | $\Theta_{C\|A}$ |
|---|---|---|
| true | true | .8 |
| true | false | .2 |
| false | true | .1 |
| false | false | .9 |

| $B$ | $C$ | $D$ | $\Theta_{D\|BC}$ |
|---|---|---|---|
| true | true | true | .95 |
| true | true | false | .05 |
| true | false | true | .9 |
| true | false | false | .1 |
| false | true | true | .8 |
| false | true | false | .2 |
| false | false | true | 0 |
| false | false | false | 1 |

| $C$ | $E$ | $\Theta_{E\|C}$ |
|---|---|---|
| true | true | .7 |
| true | false | .3 |
| false | true | 0 |
| false | false | 1 |

# Finding MAP

## Algorithm *BE-map*

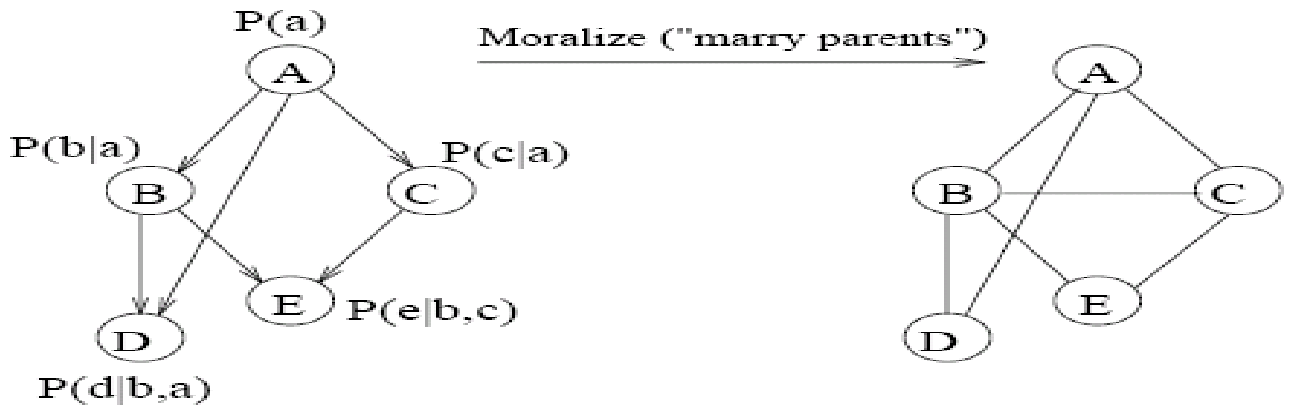


$\sum$ ***and max :***

$$MPE = \max_{a,c} P(a)P(c|a)\sum_{e,d,b} P(b|a)P(d|a,b)P(e|b,c)$$

# Finding the MAP

(An optimization task)



Variables $A$ and $B$ are the hypothesis variables.

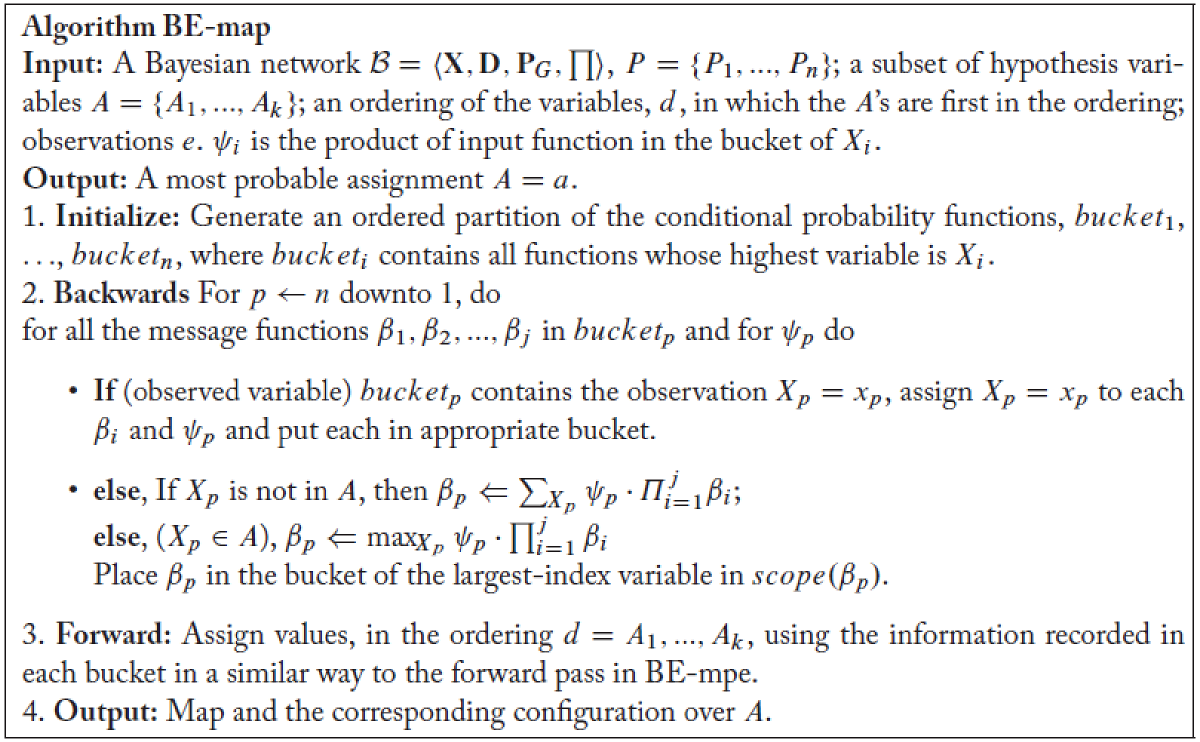
**Ordering:** $a,\ b,\ c,\ d,\ e$

$max_{a,b} P(a,b,e=0) = \max_{a,b} \sum_{c,d,e=0} P(a,b,c,d,e)$
$= \max_a P(a) \max_b P(b|a) \sum_c P(c|a) \sum_d P(d|b,a)$
$\sum_{e=0} P(e|b,c)$

**Ordering:** $a,\ e,\ d,\ c,\ b$ .... illegal ordering

$\max_{a,b} P(a,e,e=0) = \max_{a,b} \sum P(a,b,c,d,e)$
$\max_{a,b} P(a,b,e=0) = \max_a P(a) \max_b P(b|a) \sum_d \cdot$
$\max_c P(c|a) P(d|a,b) P(e=0|b,c)$

# Algorithm BE-map

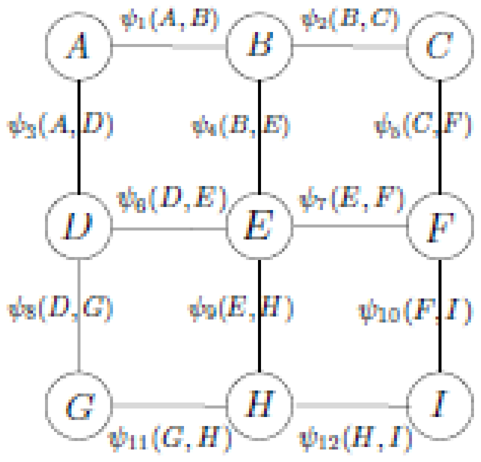


```
Algorithm BE-map
Input: A Bayesian network $\mathcal{B} = \langle \mathbf{X}, \mathbf{D}, \mathbf{P}_G, \prod \rangle$, $P = \{P_1, ..., P_n\}$; a subset of hypothesis variables $A = \{A_1, ..., A_k\}$; an ordering of the variables, $d$, in which the $A$'s are first in the ordering; observations $e$. $\psi_i$ is the product of input function in the bucket of $X_i$.
Output: A most probable assignment $A = a$.
1. Initialize: Generate an ordered partition of the conditional probability functions, $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all functions whose highest variable is $X_i$.
2. Backwards For $p \leftarrow n$ downto 1, do
for all the message functions $\beta_1, \beta_2, ..., \beta_j$ in $bucket_p$ and for $\psi_p$ do
    • If (observed variable) $bucket_p$ contains the observation $X_p = x_p$, assign $X_p = x_p$ to each $\beta_i$ and $\psi_p$ and put each in appropriate bucket.
    • else, If $X_p$ is not in $A$, then $\beta_p \Leftarrow \sum_{X_p} \psi_p \cdot \Pi_{i=1}^{j} \beta_i$;
      else, $(X_p \in A)$, $\beta_p \Leftarrow \max_{X_p} \psi_p \cdot \prod_{i=1}^{j} \beta_i$
      Place $\beta_p$ in the bucket of the largest-index variable in $scope(\beta_p)$.
3. Forward: Assign values, in the ordering $d = A_1, ..., A_k$, using the information recorded in each bucket in a similar way to the forward pass in BE-mpe.
4. Output: Map and the corresponding configuration over $A$.
```

**Variable ordering:**
**Restricted: Max buckets should**
**Be processed after sum buckets**

**Theorem 4.16** *Algorithm BE-map is complete for the map task for orderings started by the hypothesis variables. Its time and space complexity are $O(r \cdot k^{w^*_{\mathbf{E}}(d)+1})$ and $O(n \cdot k^{w^*_{\mathbf{E}}(d)})$, respectively, where $n$ is the number of variables in graph, $k$ bounds the domain size and $w^*_{\mathbf{E}}(d)$ is the conditioned induced width of its moral graph along $d$, relative to evidence variables $\mathbf{E}$. (Prove as an exercise.)* □

## BE for Markov networks queries



(a)

| $D$ | $E$ | $\psi_6(D,E)$ |
|---|---|---|
| 0 | 0 | 20.2 |
| 0 | 1 | 12 |
| 1 | 0 | 23.4 |
| 1 | 1 | 11.7 |

(b)

## Complexity of bucket elimination

**Theorem**
Given a belief network having $n$ variables, observations $e$, the complexity of elim-mpe, elim-bel, elim-map along $d$, is time and space

O(nexp(w*+1)) and O(n exp(w*)), respectively

where $w*(d)$ is the induced width of the moral graph whose edges connecting evidence to earlier nodes, were deleted.

More accurately: O(r exp(w*(d)) where r is the number of cpts.
For Bayesian networks r=n. For Markov networks?

## Finding Small Induced-Width
(Dechter 3.4-3.5)

- NP-complete
- A tree has induced-width of ?
- Greedy algorithms:
  - Min width
  - Min induced-width
  - Max-cardinality and chordal graphs
  - Fill-in (thought as the best)
  - See anytime min-width (Gogate and Dechter)

## Type of graphs

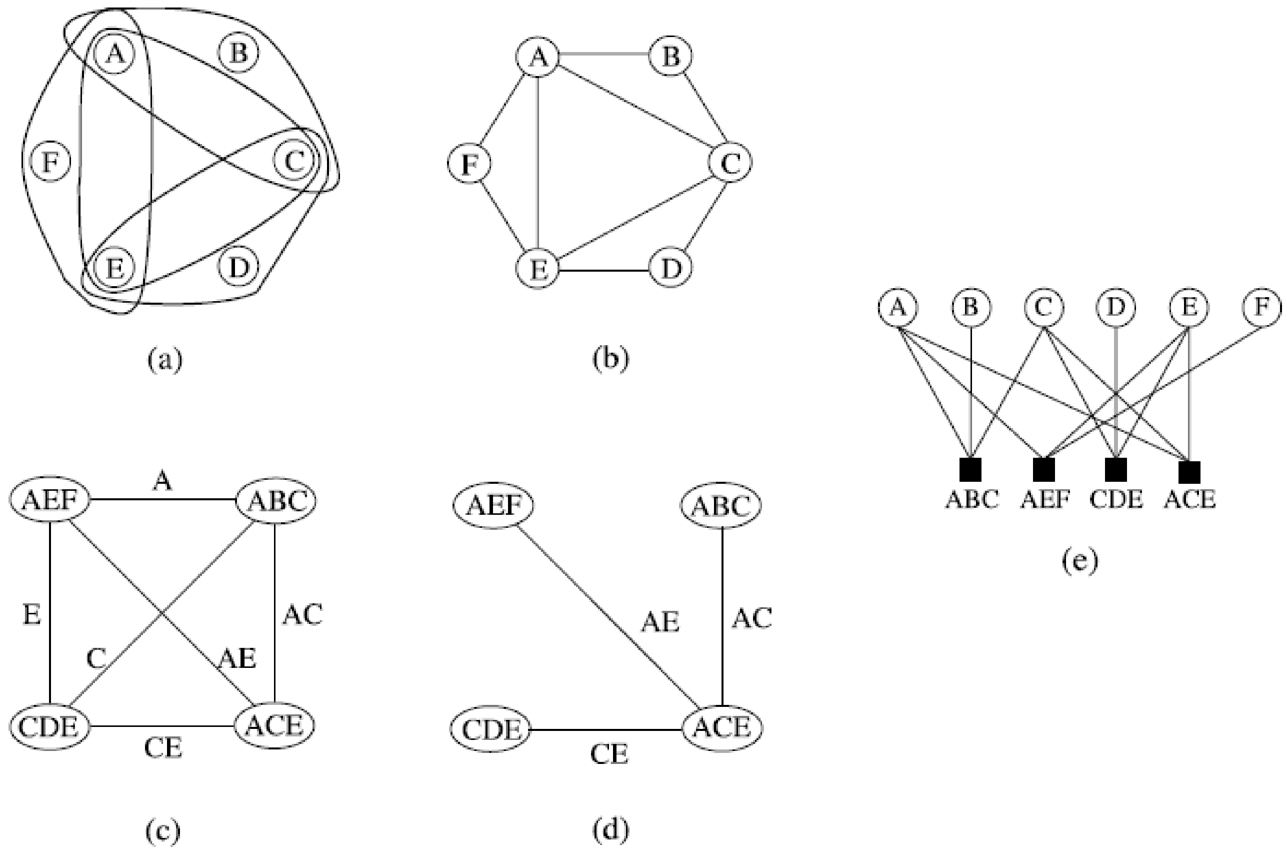


Figure 5.1: (a)Hyper, (b)Primal, (c)Dual and (d)Join-tree of a graphical model having scopes ABC, AEF, CDE and ACE. (e) the factor graph

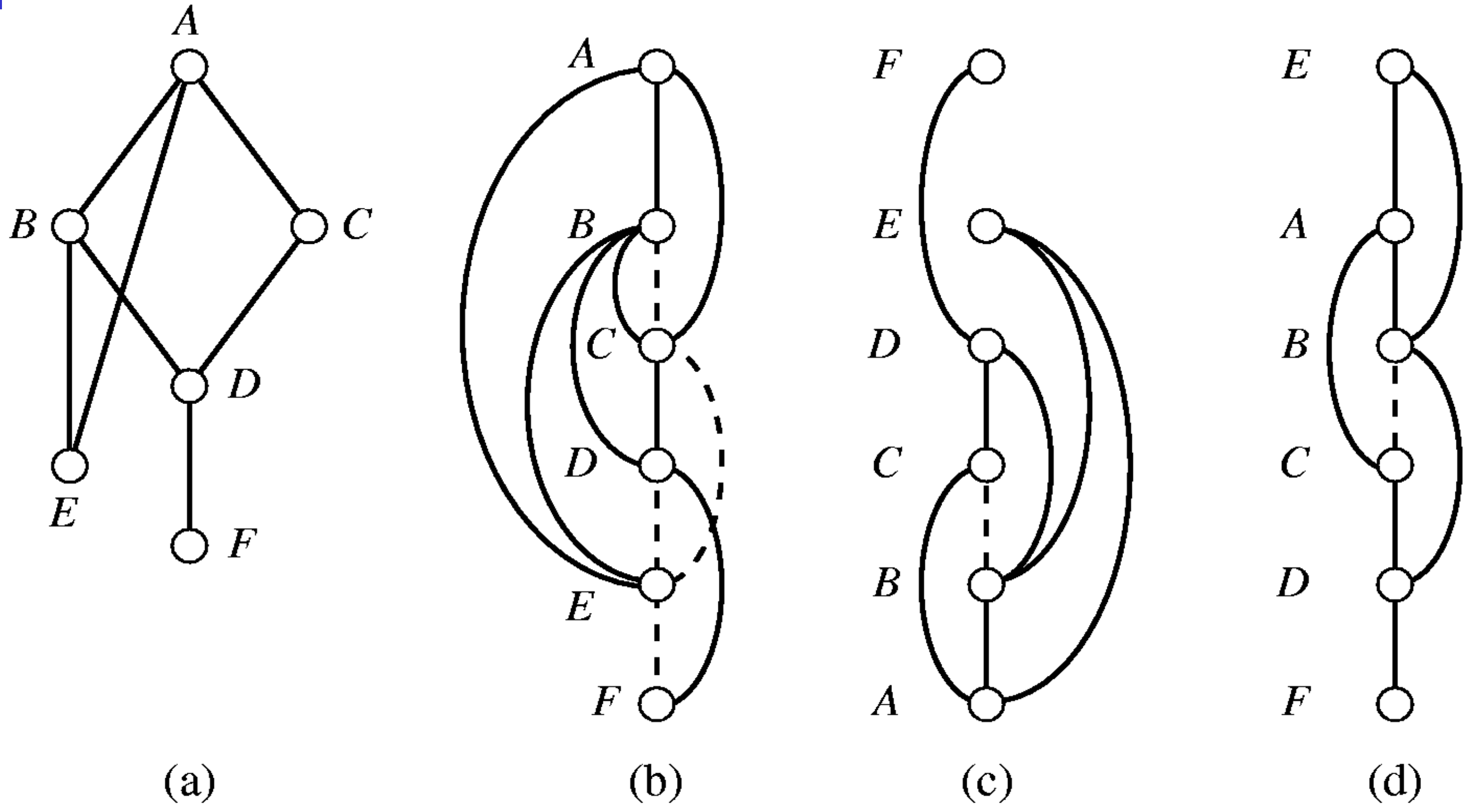
## The induced width

**Definition 5.2.1 (width)** *Given an undirected graph $G = (V, E)$, an* ordered graph *is a pair $(G, d)$, where $V = \{v_1, ..., v_n\}$ is the set of nodes, $E$ is a set of arcs over $V$, and $d = (v_1, ..., v_n)$ is an ordering of the nodes. The nodes adjacent to $v$ that precede it in the ordering are called its* parents. *The* width of a node *in an ordered graph is its number of parents. The* width of an ordering *$d$ of $G$, denoted $w_d(G)$ (or $w_d$ for short) is the maximum width over all nodes. The* width of a graph *is the minimum width over all the orderings of the graph.*

**Definition 5.2.3 (induced width)** *The* induced width of an ordered graph *$(G, d)$, denoted $w^*{}_d$), is the width of the* induced ordered graph *along $d$ obtained as follows: nodes are processed from last to first; when node $v$ is processed, all its parents are connected. The* **induced width** *of a graph, denoted by $w^*$, is the minimal induced width over all its orderings. Formally*

$$w^*(G) = \min_{d \in orderings} w^*{}_d(G)$$

# Different Induced-graphs

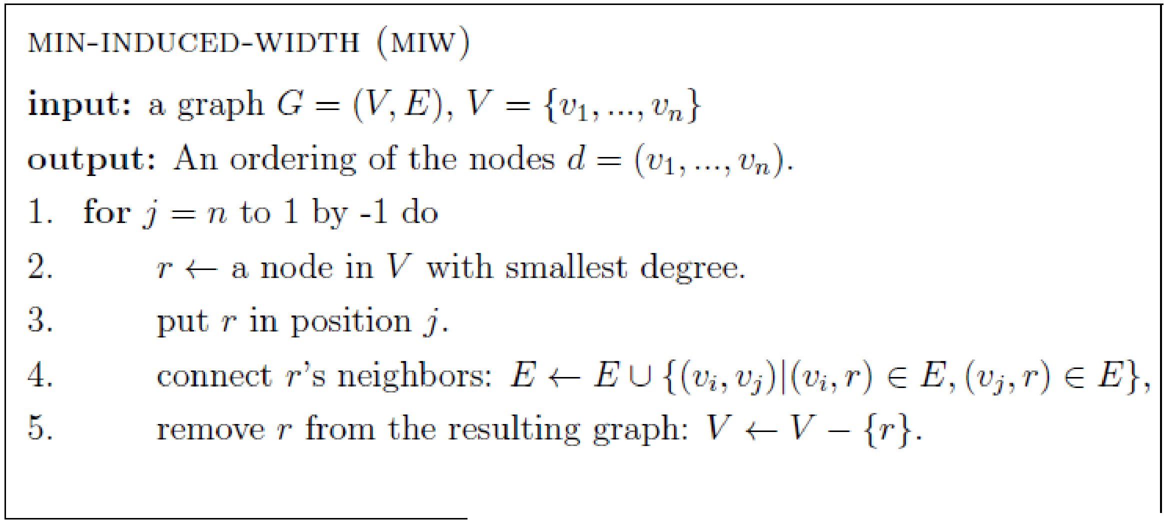


# Min-Width Ordering

```
MIN-WIDTH (MW)
input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: A min-width ordering of the nodes $d = (v_1, ..., v_n)$.
1.  for $j = n$ to 1 by -1 do
2.      $r \leftarrow$ a node in $G$ with smallest degree.
3.      put $r$ in position $j$ and $G \leftarrow G - r$.
        (Delete from $V$ node $r$ and from $E$ all its adjacent edges)
4.  endfor
```

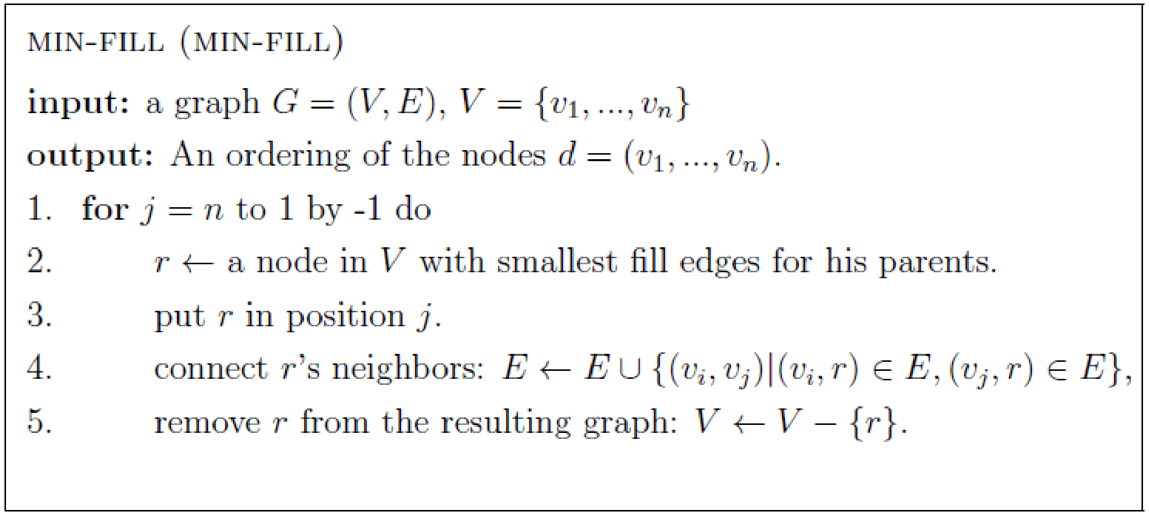
**Proposition:** (Freuder 1982) algorithm min-width finds a min-width ordering of a graph. Complexity O(|E|)

## Greedy Orderings Heuristics



```
MIN-INDUCED-WIDTH (MIW)
input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1. for $j = n$ to 1 by -1 do
2.         $r \leftarrow$ a node in $V$ with smallest degree.
3.         put $r$ in position $j$.
4.         connect $r$'s neighbors: $E \leftarrow E \cup \{(v_i, v_j) | (v_i, r) \in E, (v_j, r) \in E\}$,
5.         remove $r$ from the resulting graph: $V \leftarrow V - \{r\}$.
```

**Theorem:** A graph is a tree iff it has both width and induced-width of 1.

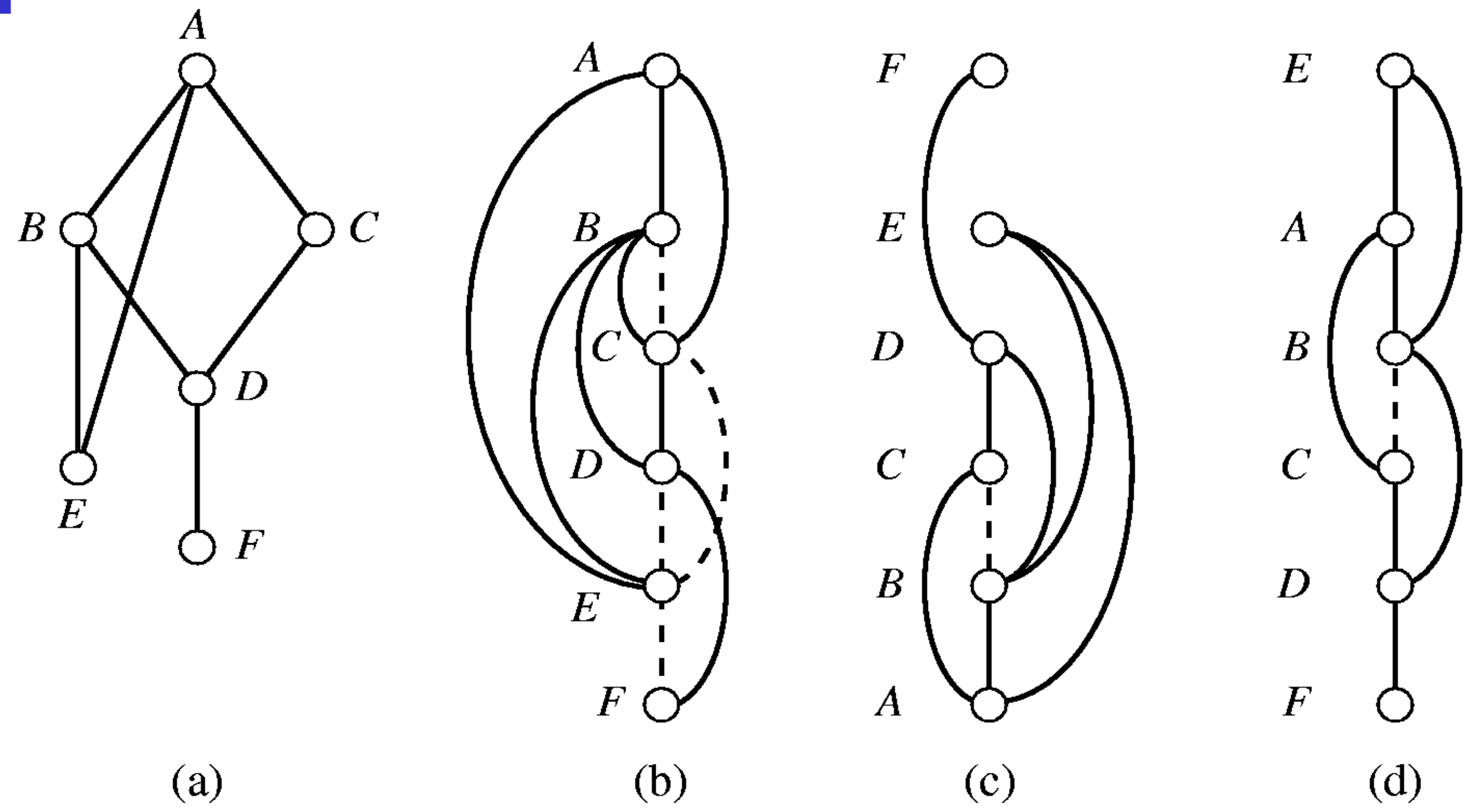


```
MIN-FILL (MIN-FILL)
input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1. for $j = n$ to 1 by -1 do
2.         $r \leftarrow$ a node in $V$ with smallest fill edges for his parents.
3.         put $r$ in position $j$.
4.         connect $r$'s neighbors: $E \leftarrow E \cup \{(v_i, v_j) | (v_i, r) \in E, (v_j, r) \in E\}$,
5.         remove $r$ from the resulting graph: $V \leftarrow V - \{r\}$.
```

Complexity?

O(n^3)

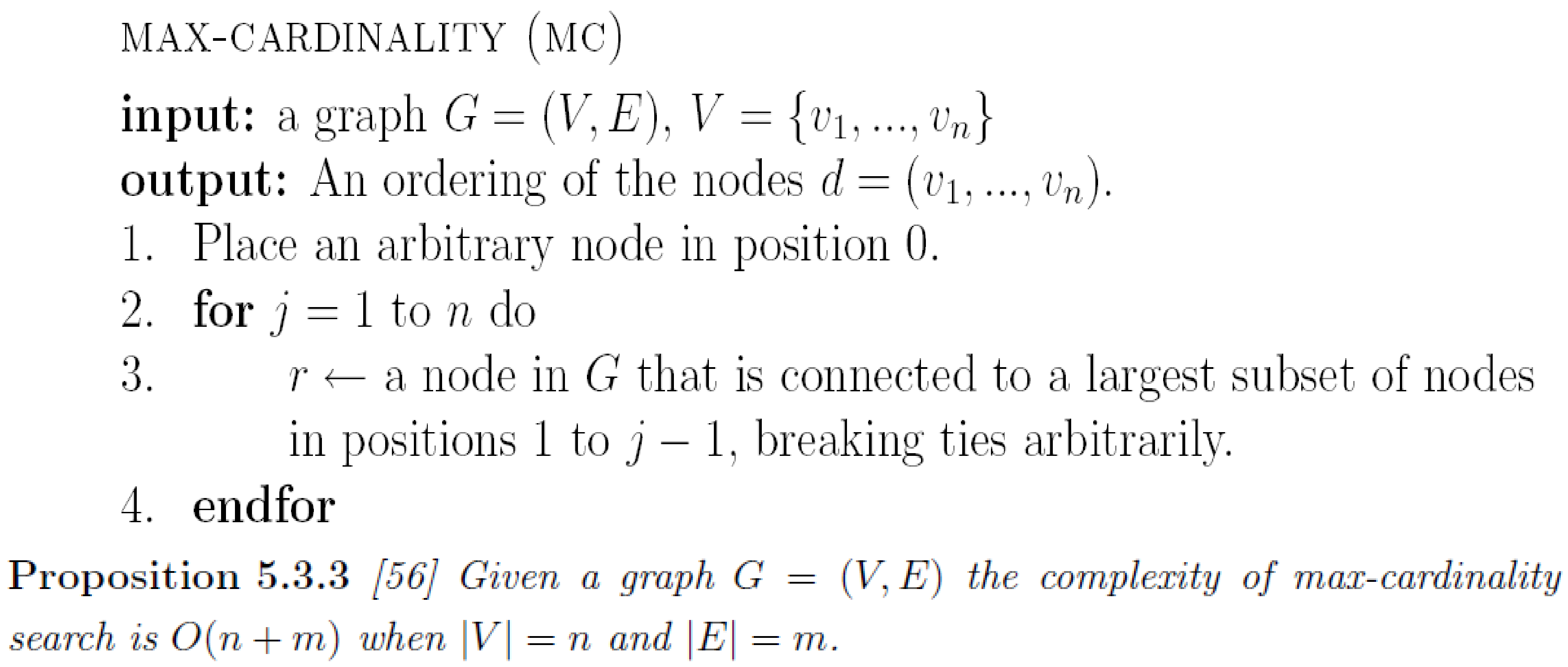
## Different Induced-Graphs



## Induced-width for chordal graphs

- **Definition:** A graph is chordal if every cycle of length at least 4 has a chord
- Finding w* over chordal graph is easy using the **max-cardinality ordering:** order vertices from 1 to n, always assigning the next number to the node connected to a largest set of previously numbered nodes. Lets d be such an ordering
- A graph along max-cardinality order has no fill-in edges iff it is chordal.
- On chordal graphs width=induced-width.

# Max-cardinality ordering



MAX-CARDINALITY (MC)

```
input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1.  Place an arbitrary node in position 0.
2.  for $j = 1$ to $n$ do
3.          $r \leftarrow$ a node in $G$ that is connected to a largest subset of nodes
            in positions 1 to $j - 1$, breaking ties arbitrarily.
4.  endfor
```

**Proposition 5.3.3** *[56] Given a graph $G = (V, E)$ the complexity of max-cardinality search is $O(n + m)$ when $|V| = n$ and $|E| = m$.*

What is the complexity of min-fill? Min-induced-width? $O(n^3)$

## K-trees

**Definition 5.3.4 (k-trees)** *A subclass of chordal graphs are k-trees. A k-tree is a chordal graph whose maximal cliques are of size $k+1$, and it can be defined recursively as follows: (1) A complete graph with $k$ vertices is a k-tree. (2) A k-tree with $r$ vertices can be extended to $r+1$ vertices by connecting the new vertex to all the vertices in any clique of size $k$. A partial k-tree is a k-tree having some of its arcs removed. Namely it will clique of size smaller than $k$.*

## Which greedy algorithm is best?

- MinFill, prefers a node who add the least number of fill-in arcs.
- Empirically, fill-in is the best among the greedy algorithms (MW,MIW,MF,MC)
- Complexity of greedy orderings?
  - MW is $O(n^2)$...maybe $O(nlogn + m\ \ )$?
  - MIW: O($O(n^3)$,
  - MF ($O(n^3)$,
  - MC is O(m+n), m edges.

# Recent work in my group

- **Vibhav Gogate and Rina Dechter.** "A Complete Anytime Algorithm for Treewidth". *In UAI 2004.*
- **Andrew E. Gelfand, Kalev Kask, and Rina Dechter.** "Stopping Rules for Randomized Greedy Triangulation Schemes" in *Proceedings of AAAI 2011.*
- Potential project

## Mixed Networks

- Augmenting Probabilistic networks with constraints because:
  - Some information in the world is deterministic and undirected (X not-eq Y)
  - Some queries are complex or evidence are complex (cnfs)
- Queries are probabilistic queries

## Mixed Beliefs and Constraints

- Assume the CN is a cnf formula
- Queries over hybrid network:
- Complex evidence structure

$$\varphi = (G \vee D) \wedge (\neg D \vee B)$$

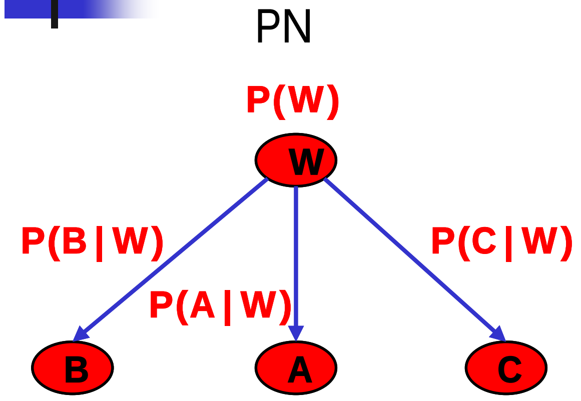
$$P(\varphi) = ?$$

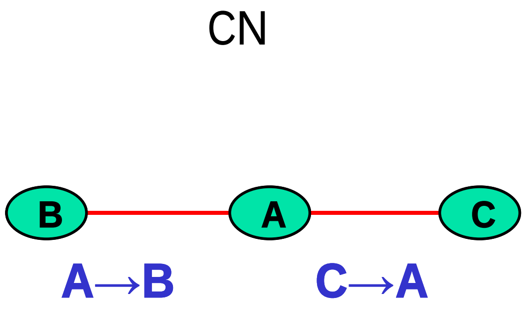
$$P(\bar{x} \mid \varphi) = ?$$

$$P(x_1 \mid \varphi) = ?$$

- All reduce to CNF queries over a Belief network:
  - CPE (CNF probability evaluation): Given a belief network, and a cnf, find its probability.

## Party example again





**Query:**
*Is it likely that Chris goes to the party if Becky does not but the weather is bad?*

$$P(C,\neg B \mid w = bad, A \to B, C \to A)$$

Semantics?

Algorithms?

## Bucket Elimination for Mixed networks

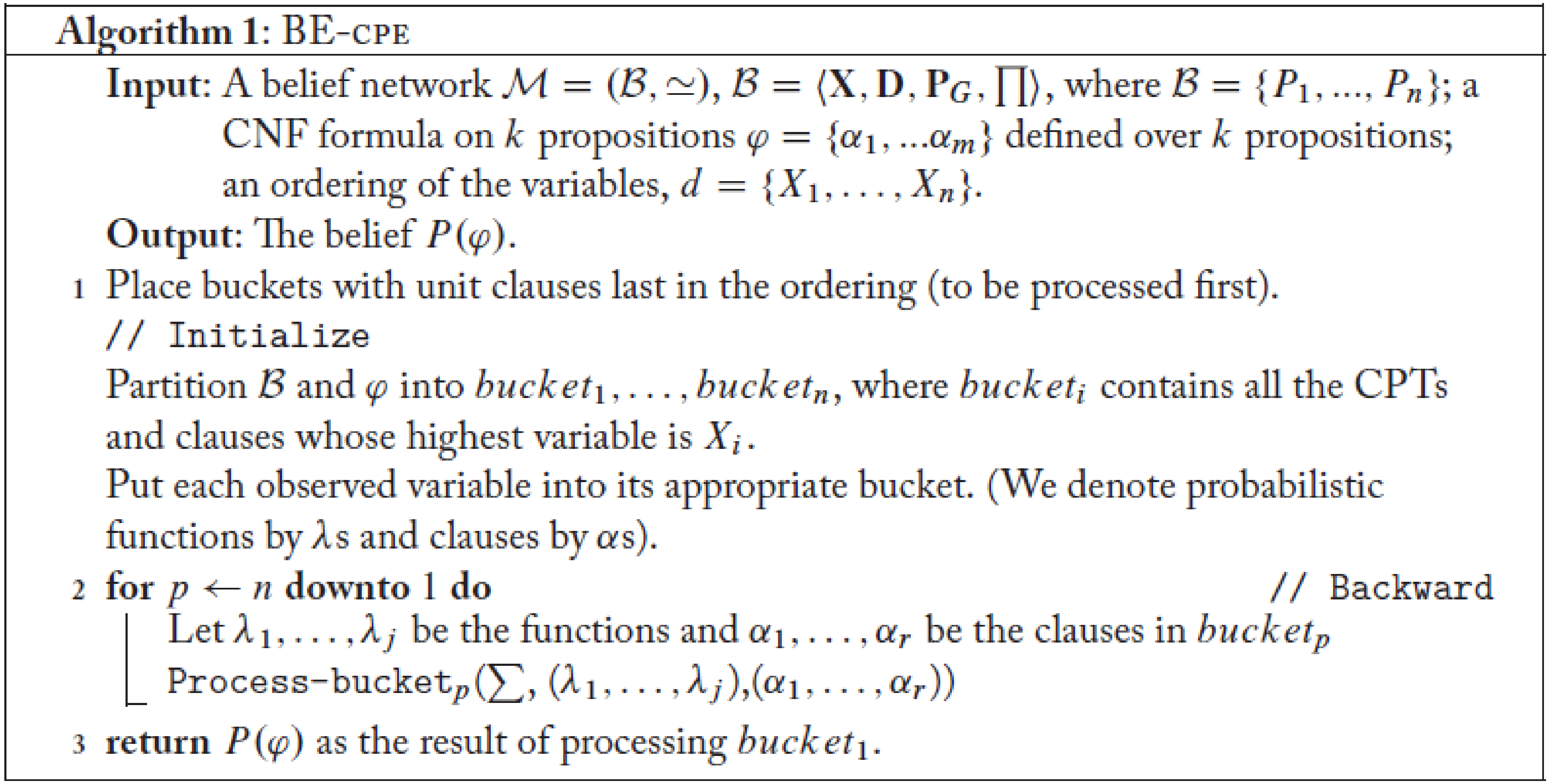
The CPE query

$$P_{\mathcal{B}}(\varphi) = \sum_{\mathbf{x}_\varphi \in Mod(\varphi)} P(\mathbf{x}_\varphi)$$

Using the belief network product form we get:

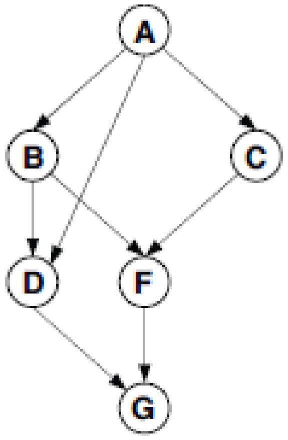
$$P_{\mathcal{B}}(\varphi) = \sum_{\{\mathbf{x} | \mathbf{x}_\varphi \in Mod(\varphi)\}} \prod_{i=1}^{n} P(x_i | \mathbf{x}_{pa_i}).$$

P((C → B) and P(A → C))

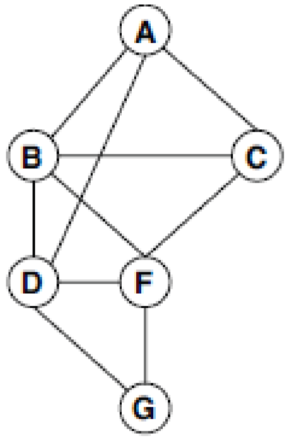


```
Algorithm 1: BE-CPE
    Input: A belief network $\mathcal{M} = (\mathcal{B}, \simeq)$, $\mathcal{B} = \langle \mathbf{X}, \mathbf{D}, \mathbf{P}_G, \prod \rangle$, where $\mathcal{B} = \{P_1, ..., P_n\}$; a
           CNF formula on $k$ propositions $\varphi = \{\alpha_1, ...\alpha_m\}$ defined over $k$ propositions;
           an ordering of the variables, $d = \{X_1, \ldots, X_n\}$.
    Output: The belief $P(\varphi)$.
  1 Place buckets with unit clauses last in the ordering (to be processed first).
    // Initialize
    Partition $\mathcal{B}$ and $\varphi$ into $bucket_1, \ldots, bucket_n$, where $bucket_i$ contains all the CPTs
    and clauses whose highest variable is $X_i$.
    Put each observed variable into its appropriate bucket. (We denote probabilistic
    functions by $\lambda$s and clauses by $\alpha$s).
  2 for $p \leftarrow n$ downto 1 do                                          // Backward
        Let $\lambda_1, \ldots, \lambda_j$ be the functions and $\alpha_1, \ldots, \alpha_r$ be the clauses in $bucket_p$
        Process-bucket$_p$($\sum$, $(\lambda_1, \ldots, \lambda_j)$,$(\alpha_1, \ldots, \alpha_r)$)
  3 return $P(\varphi)$ as the result of processing $bucket_1$.
```

```
Procedure Process-bucket_p($\sum$, $(\lambda_1, \ldots, \lambda_j)$,$(\alpha_1, \ldots, \alpha_r)$ ).
    if bucket_p contains evidence $X_p = x_p$ then
        1. Assign $X_p = x_p$ to each $\lambda_i$ and put each resulting function in the bucket of its latest variable
        2. Resolve each $\alpha_i$ with the unit clause, put non-tautology resolvents in the buckets of their latest variable and move any bucket with unit clause to top of processing
    else
        $\lambda_p \leftarrow \sum_{\{x_p | \mathbf{x}_{U_p} \in Mod(\alpha_1,\ldots,\alpha_r)\}} \prod_{i=1}^{j} \lambda_i$
        Add $\lambda_p$ to the bucket of the latest variable in $S_p$, where
        $S_p = scope(\lambda_1, ..., \lambda_j, \alpha_1, ..., \alpha_r)$, $U_p = scope(\alpha_1, ..., \alpha_r)$.
```

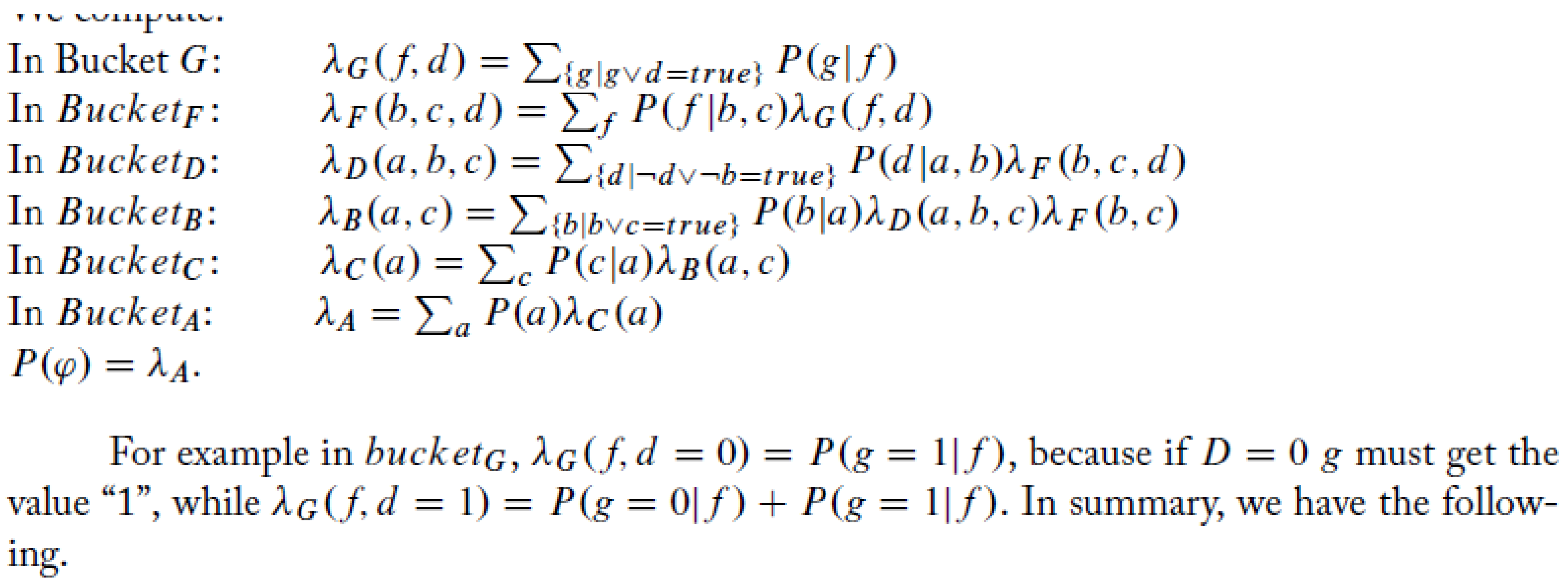


(a) Directed acyclic graph



(b) Moral graph

# Processing  Mixed Buckets



We compute:

In Bucket $G$: $\lambda_G(f,d) = \sum_{\{g|g \vee d = true\}} P(g|f)$

In $Bucket_F$: $\lambda_F(b,c,d) = \sum_f P(f|b,c)\lambda_G(f,d)$

In $Bucket_D$: $\lambda_D(a,b,c) = \sum_{\{d|\neg d \vee \neg b = true\}} P(d|a,b)\lambda_F(b,c,d)$

In $Bucket_B$: $\lambda_B(a,c) = \sum_{\{b|b \vee c = true\}} P(b|a)\lambda_D(a,b,c)\lambda_F(b,c)$

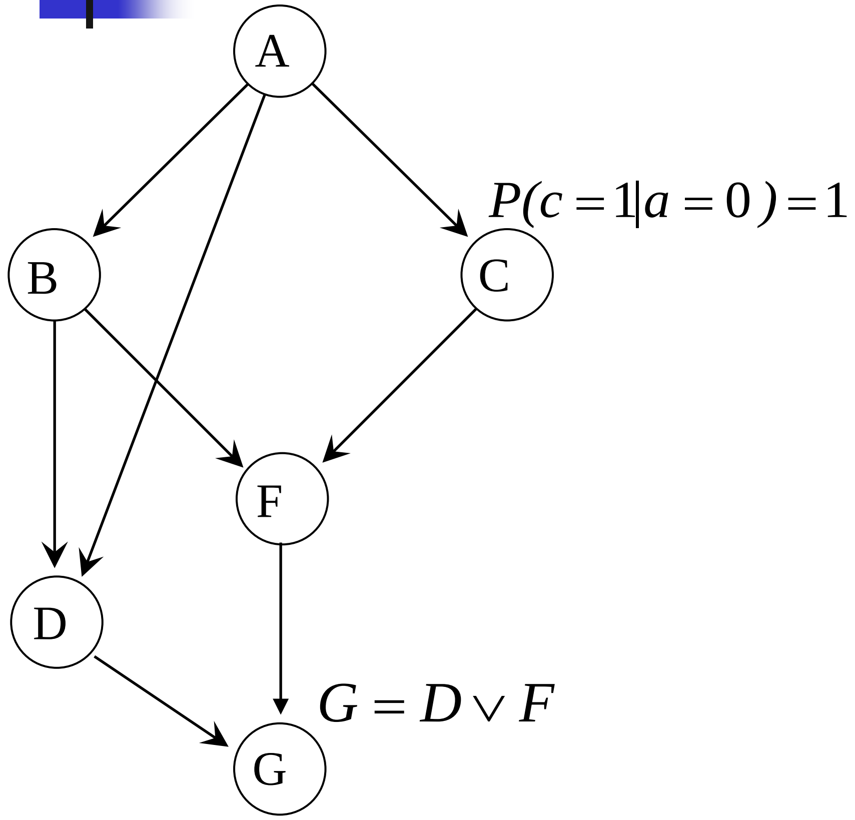
In $Bucket_C$: $\lambda_C(a) = \sum_c P(c|a)\lambda_B(a,c)$

In $Bucket_A$: $\lambda_A = \sum_a P(a)\lambda_C(a)$

$P(\varphi) = \lambda_A$.

For example in $bucket_G$, $\lambda_G(f, d = 0) = P(g = 1|f)$, because if $D = 0$ $g$ must get the value "1", while $\lambda_G(f, d = 1) = P(g = 0|f) + P(g = 1|f)$. In summary, we have the following.

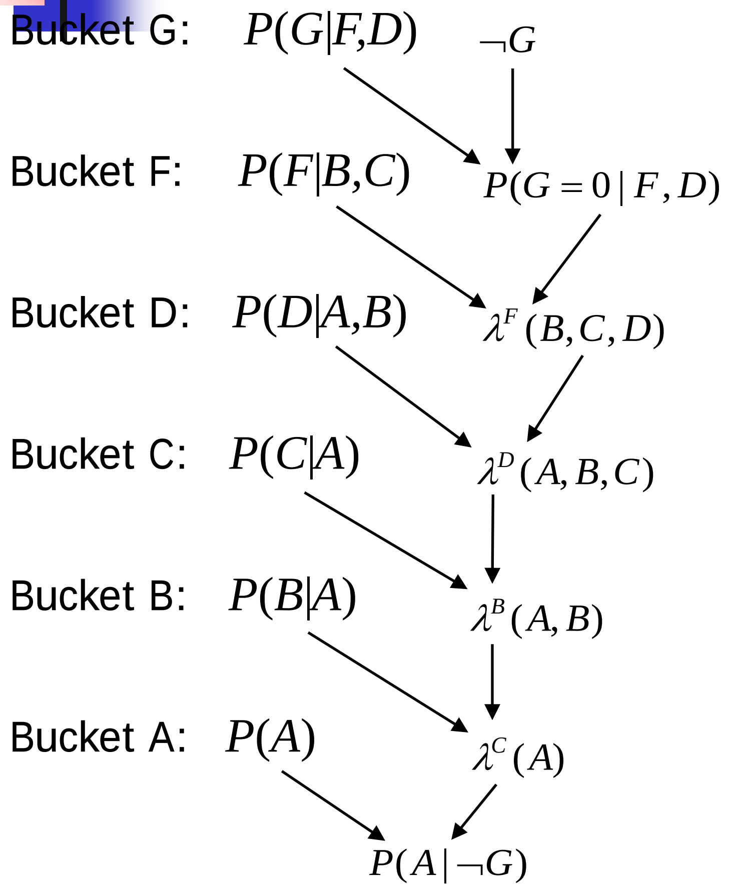
# A Hybrid Belief Network



$$\begin{tikzcd}
\text{Bucket G:} & P(G|F,D) \arrow[dr] & \neg G \arrow[d] \\
\text{Bucket F:} & P(F|B,C) \arrow[dr] & P(G=0\,|\,F,D) \arrow[d] \\
\text{Bucket D:} & P(D|A,B) \arrow[dr] & \lambda^F(B,C,D) \arrow[d] \\
\text{Bucket C:} & P(C|A) \arrow[dr] & \lambda^D(A,B,C) \arrow[d] \\
\text{Bucket B:} & P(B|A) \arrow[dr] & \lambda^B(A,B) \arrow[d] \\
\text{Bucket A:} & P(A) \arrow[dr] & \lambda^C(A) \arrow[d] \\
 & & P(A\,|\,\neg G)
\end{tikzcd}$$

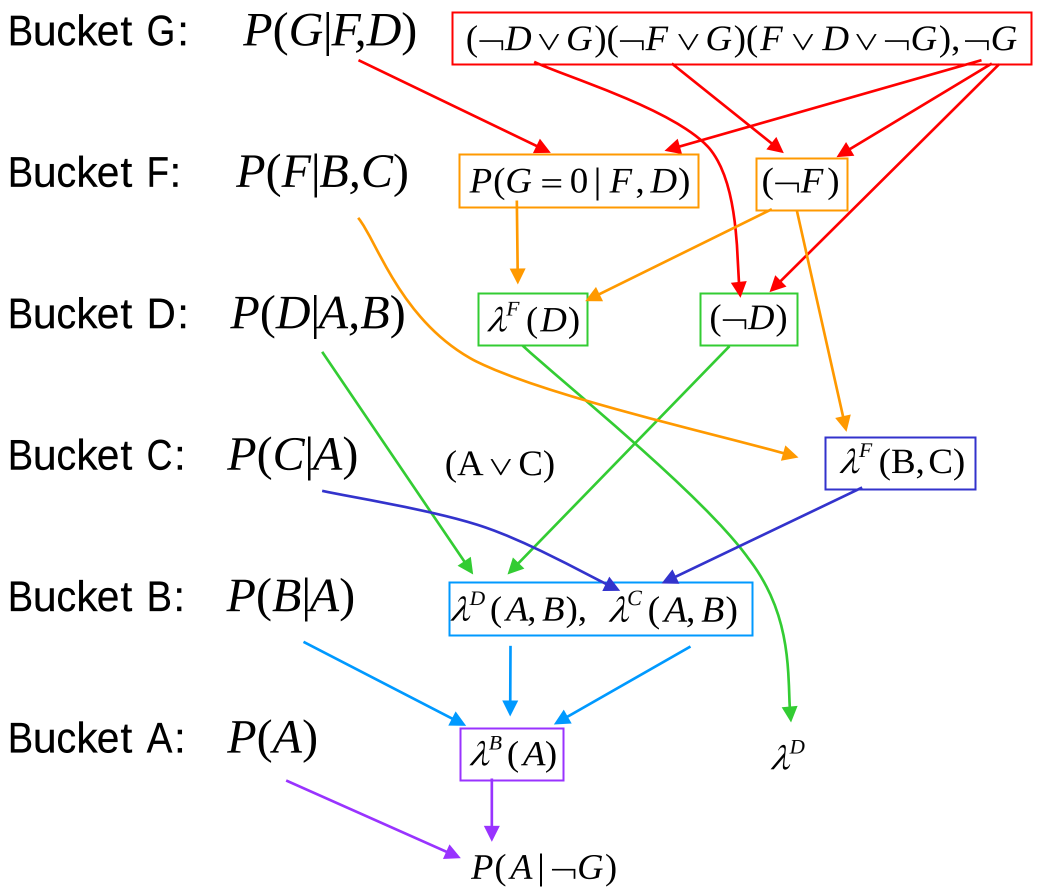
Belief network P(g,f,d,c,b,a)
=P(g|f,d)P(f|c,b)P(d|b,a)P(b|a)P(c|a)P(a)

# Variable elimination for a mixed network:



Bucket G: $P(G|F,D)$ $\neg G$
Bucket F: $P(F|B,C)$ $P(G=0|F,D)$
Bucket D: $P(D|A,B)$ $\lambda^F(B,C,D)$
Bucket C: $P(C|A)$ $\lambda^D(A,B,C)$
Bucket B: $P(B|A)$ $\lambda^B(A,B)$
Bucket A: $P(A)$ $\lambda^C(A)$
$P(A|\neg G)$

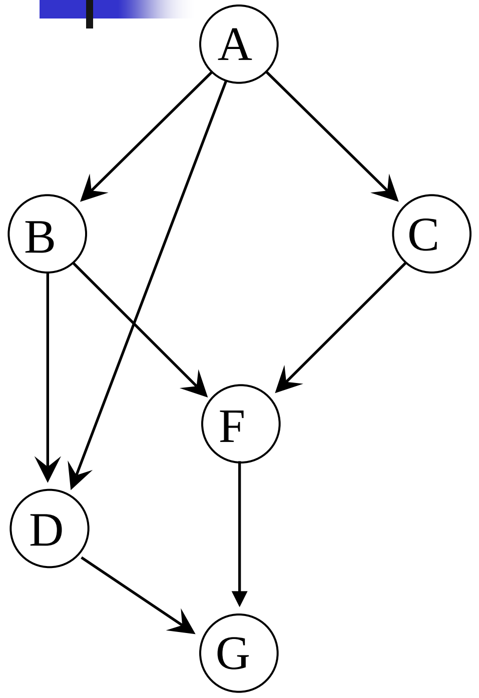
(a) regular Elim-CPE

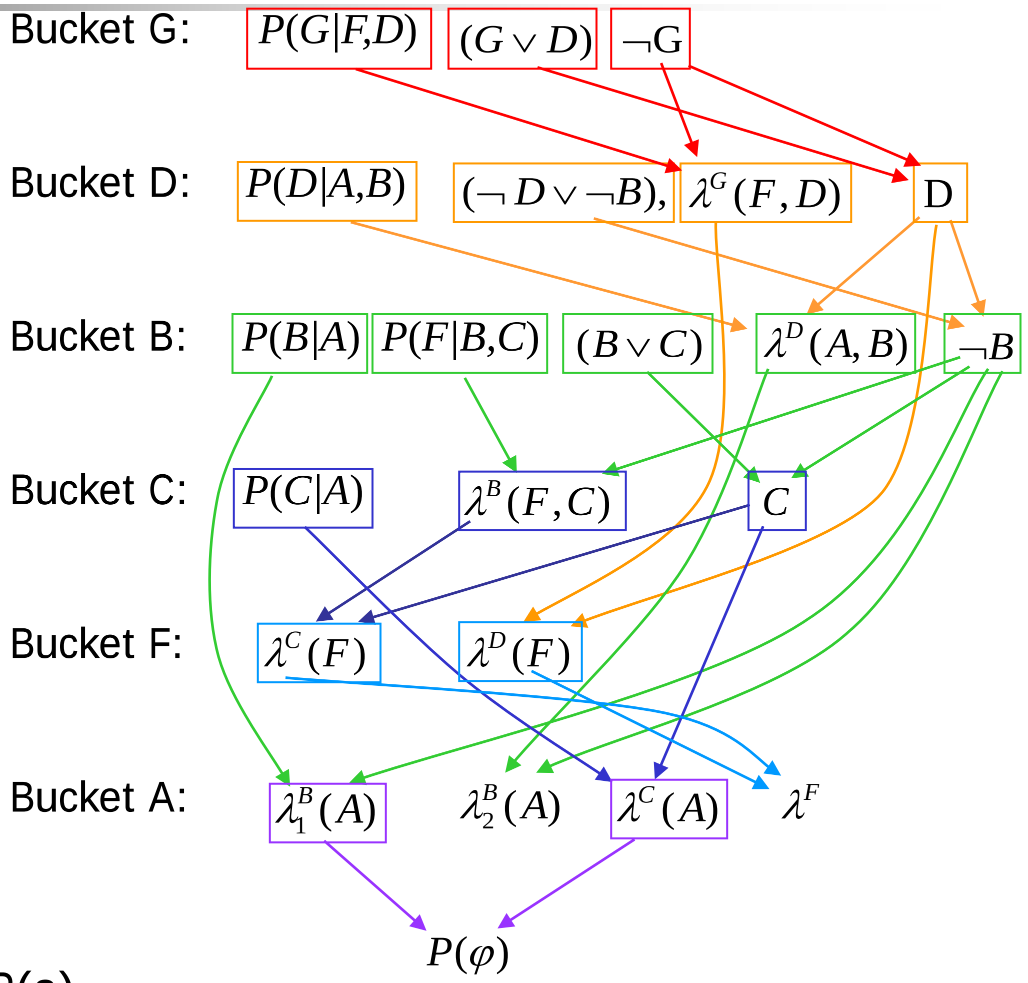


Bucket G: $P(G|F,D)$ $(\neg D \vee G)(\neg F \vee G)(F \vee D \vee \neg G), \neg G$
Bucket F: $P(F|B,C)$ $P(G=0|F,D)$ $(\neg F)$
Bucket D: $P(D|A,B)$ $\lambda^F(D)$ $(\neg D)$
Bucket C: $P(C|A)$ $(A \vee C)$ $\lambda^F(B,C)$
Bucket B: $P(B|A)$ $\lambda^D(A,B),\ \lambda^C(A,B)$
Bucket A: $P(A)$ $\lambda^B(A)$ $\lambda^D$
$P(A|\neg G)$

(b) Elim-CPE-D with clause extraction

# Trace of Elim-CPE





Belief network P(g,f,d,c,b,a)
=P(g|f,d)P(f|c,b)P(d|b,a)P(b|a)P(c|a)P(a)

## Bucket-elimination example for a mixed network

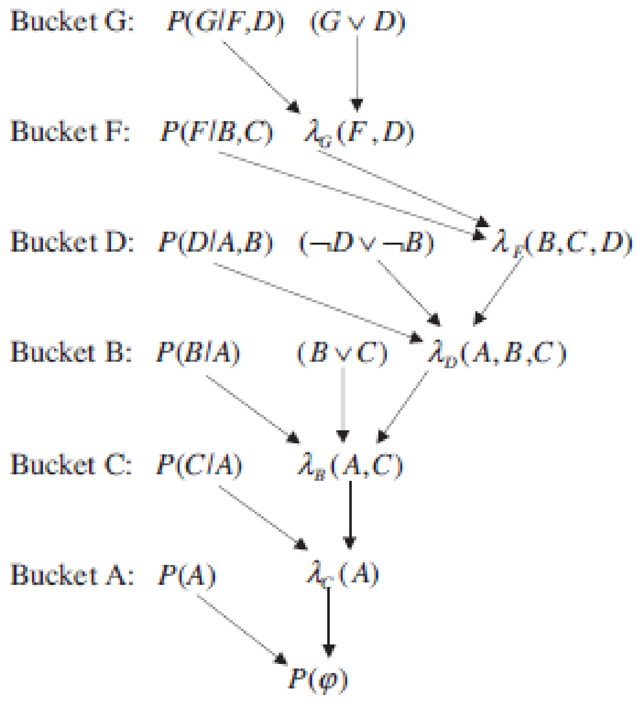

Bucket G: $P(G|F,D)$ $(G \vee D)$
Bucket F: $P(F|B,C)$ $\lambda_G(F,D)$
Bucket D: $P(D|A,B)$ $(\neg D \vee \neg B)$ $\lambda_F(B,C,D)$
Bucket B: $P(B|A)$ $(B \vee C)$ $\lambda_D(A,B,C)$
Bucket C: $P(C|A)$ $\lambda_B(A,C)$
Bucket A: $P(A)$ $\lambda_C(A)$
$P(\varphi)$

Figure 4.15: Execution of BE-CPE.

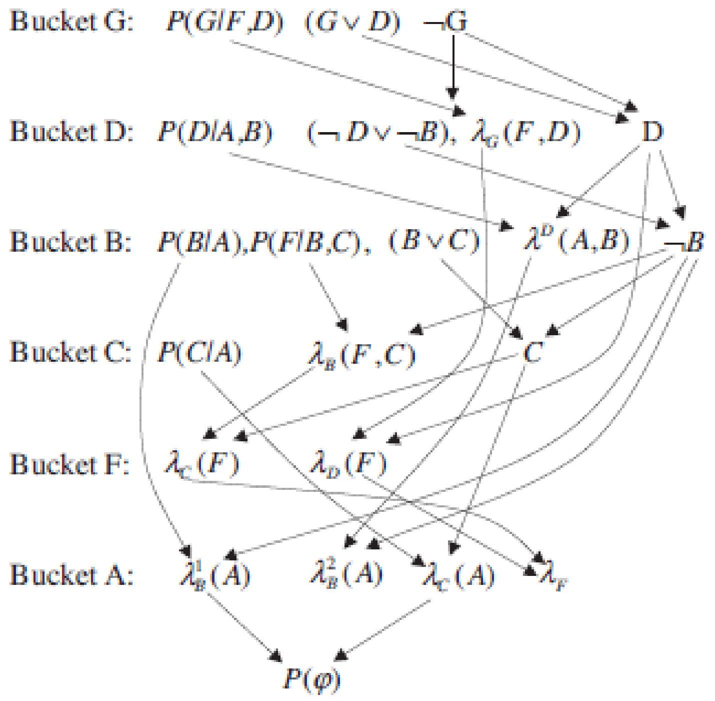
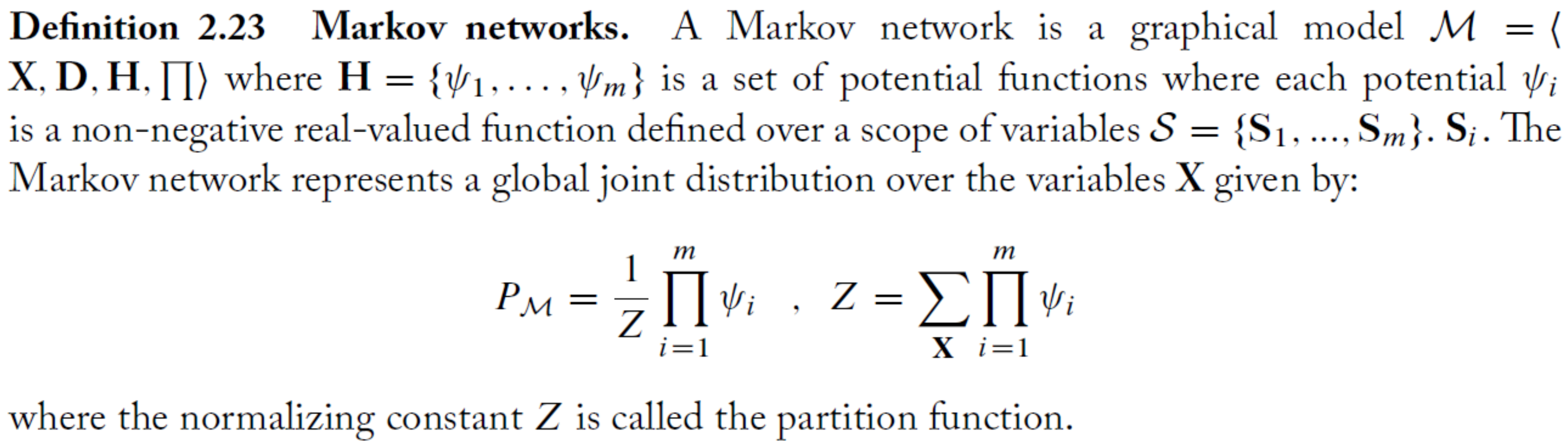

Bucket G: $P(G|F,D)$ $(G \vee D)$ $\neg G$
Bucket D: $P(D|A,B)$ $(\neg D \vee \neg B)$, $\lambda_G(F,D)$ D
Bucket B: $P(B|A), P(F|B,C)$, $(B \vee C)$ $\lambda^D(A,B)$ $\neg B$
Bucket C: $P(C|A)$ $\lambda_B(F,C)$ C
Bucket F: $\lambda_C(F)$ $\lambda_D(F)$
Bucket A: $\lambda_B^1(A)$ $\lambda_B^2(A)$ $\lambda_C(A)$ $\lambda_F$
$P(\varphi)$

Figure 4.16: Execution of BE-CPE (evidence $\neg G$).

# Markov Networks



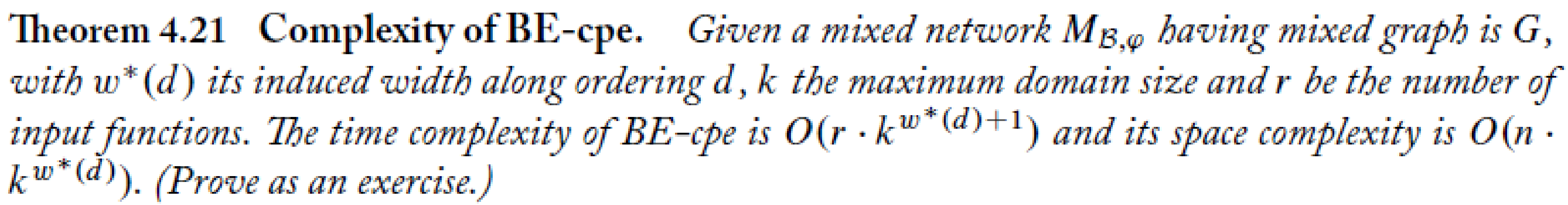
**Definition 2.23 Markov networks.** A Markov network is a graphical model $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{H}, \prod \rangle$ where $\mathbf{H} = \{\psi_1, \ldots, \psi_m\}$ is a set of potential functions where each potential $\psi_i$ is a non-negative real-valued function defined over a scope of variables $\mathcal{S} = \{\mathbf{S}_1, ..., \mathbf{S}_m\}$. $\mathbf{S}_i$. The Markov network represents a global joint distribution over the variables $\mathbf{X}$ given by:

$$P_{\mathcal{M}} = \frac{1}{Z} \prod_{i=1}^{m} \psi_i \quad , \quad Z = \sum_{\mathbf{X}} \prod_{i=1}^{m} \psi_i$$

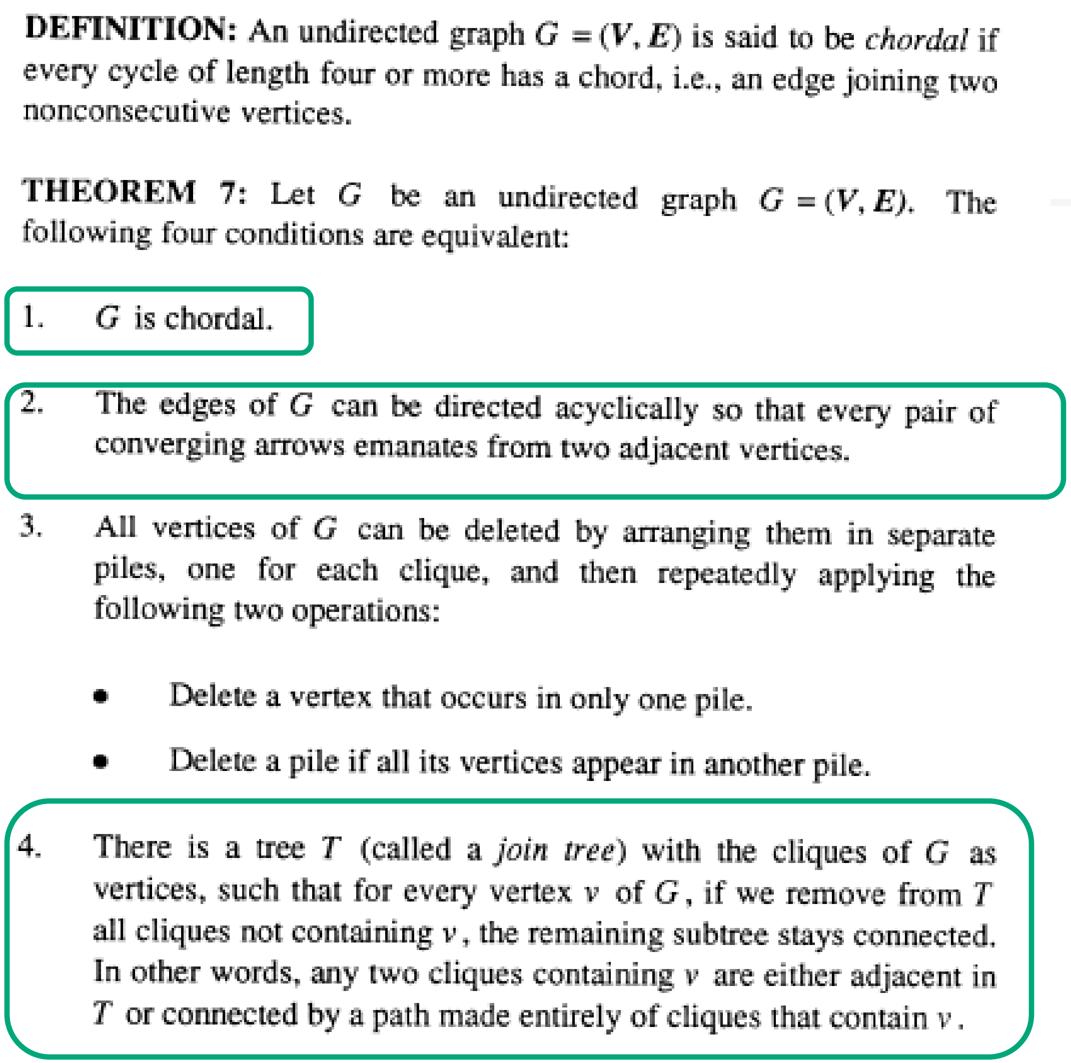
where the normalizing constant $Z$ is called the partition function.

Dechter, chapter 2

## Complexity



**Theorem 4.21 Complexity of BE-cpe.** *Given a mixed network $M_{\mathcal{B},\varphi}$ having mixed graph is $G$, with $w^*(d)$ its induced width along ordering $d$, $k$ the maximum domain size and $r$ be the number of input functions. The time complexity of BE-cpe is $O(r \cdot k^{w^*(d)+1})$ and its space complexity is $O(n \cdot k^{w^*(d)})$. (Prove as an exercise.)*



**DEFINITION:** An undirected graph $G = (V, E)$ is said to be *chordal* if every cycle of length four or more has a chord, i.e., an edge joining two nonconsecutive vertices.

**THEOREM 7:** Let $G$ be an undirected graph $G = (V, E)$. The following four conditions are equivalent:

1. $G$ is chordal.
2. The edges of $G$ can be directed acyclically so that every pair of converging arrows emanates from two adjacent vertices.
3. All vertices of $G$ can be deleted by arranging them in separate piles, one for each clique, and then repeatedly applying the following two operations:
   - Delete a vertex that occurs in only one pile.
   - Delete a pile if all its vertices appear in another pile.
4. There is a tree $T$ (called a *join tree*) with the cliques of $G$ as vertices, such that for every vertex $v$ of $G$, if we remove from $T$ all cliques not containing $v$, the remaining subtree stays connected. In other words, any two cliques containing $v$ are either adjacent in $T$ or connected by a path made entirely of cliques that contain $v$.

The running intersection property