CS 276
Probabilistic and Causal Reasoning

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Algorithmic Approach for Identification

Based on Elias Bareinboim slides

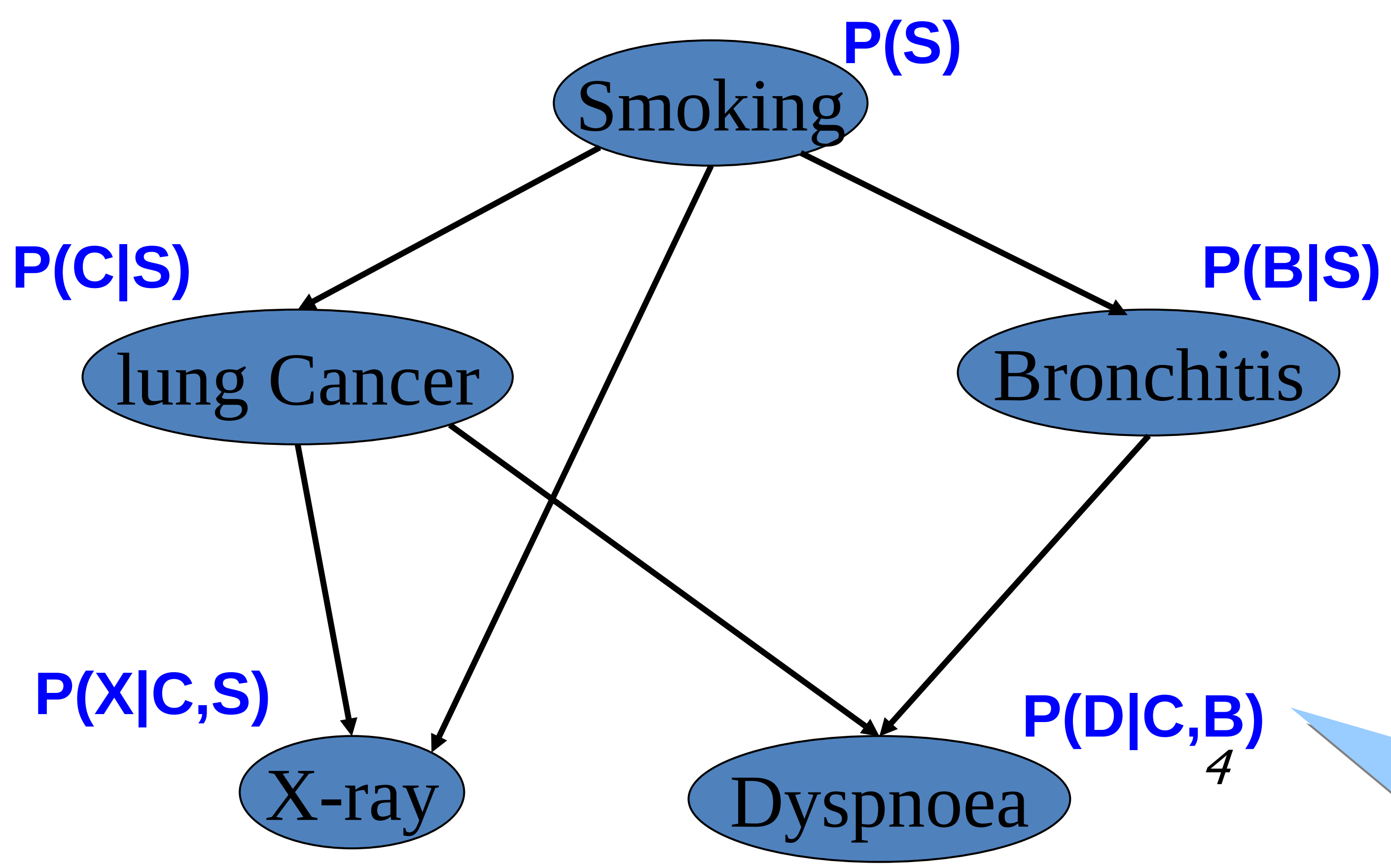
Roadmap

- Define a decomposition (factorization) of the probability distributions generated by a SCM, based on the coresponding causal diagram.
- Establish operations that allows us to identify particular components (factors) from a distribution.
- Express the target causal effect into factors and develop a systematic procedure to identify each one of them independently.

Factorizing Observational Distributions

Bayesian Networks: Example

(Pearl, 1988)



BN = (G, Θ)

CPD:

C	B	P(D|C,B)	
0	0	0.1	0.9
0	1	0.7	0.3
1	0	0.8	0.2
1	1	0.9	0.1

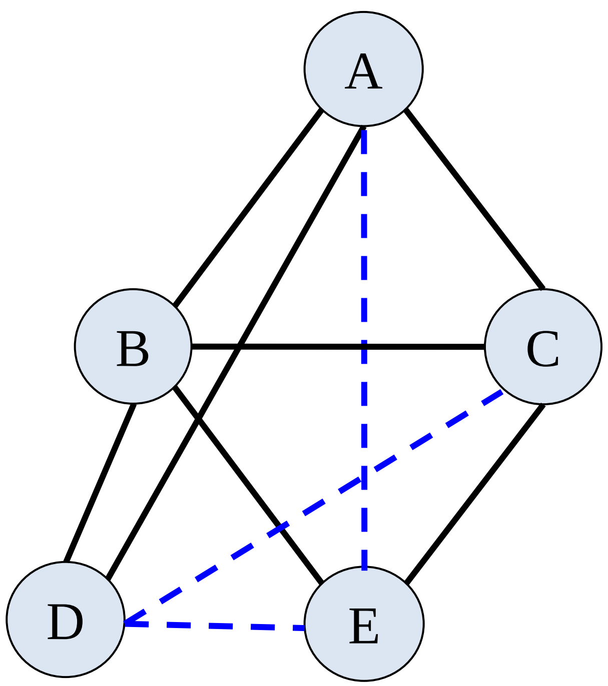
$$P(S, C, B, X, D) = P(S)\, P(C|S)\, P(B|S)\, P(X|C,S)\, P(D|C,B)$$

Belief Updating:

P (lung cancer=yes | smoking=no, dyspnoea=yes) = ?

Belief Updating

p(X | Evidence) = ?



“primal” graph

$$p(A|E=0)$$

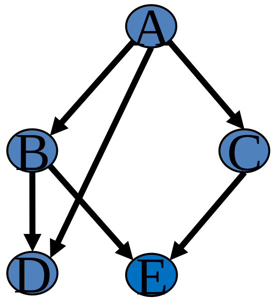
$$\propto p(A, E=0)$$

$$= \sum_{e,d,c,b} p(A)\, p(b|A)\, p(c|A)\, p(d|b,A)\, p(e|b,c)\, \mathbb{1}[e=0]$$

$$p(A) \sum_{e} \sum_{d} \sum_{c} p(c|A)\, \mathbb{1}[e=0] \underbrace{\sum_{b} p(b|A)\, p(d|b,A)\, p(e|b,c)}_{\lambda_{B \to C}(a,d,c,e)}$$

Variable Elimination

Belief Updating



Algorithm *BE-bel* [Dechter 1996]

$$p(A|E=0) = \alpha \sum_{e,d,c,b} p(A)\, p(b|A)\, p(c|A)\, p(d|A,b)\, p(e|b,c)\, \mathbb{1}[e=0]$$

$\sum_b \prod$ ← Elimination & combination operators

bucket B: $p(b|A)\ p(d|b,A)\ p(e|b,c)$

bucket C: $p(c|A)$ $\lambda_{B\to C}(A,d,c,e)$

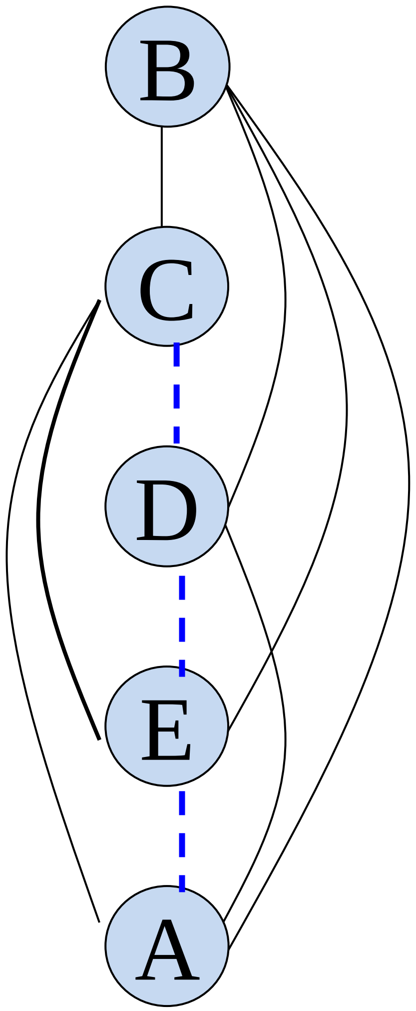
bucket D: $\lambda_{C\to D}(A,d,e)$

bucket E: $\mathbb{1}[E=0]\ \lambda_{D\to E}(A,e)$

bucket A: $p(A)$ $\lambda_{E\to A}(A)$

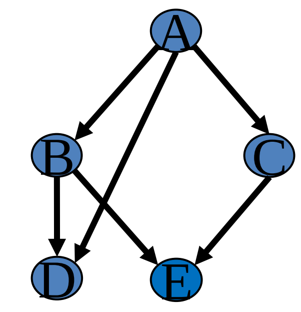
$p(E=0)$

W*=4
"induced width"
(max clique size)



$p(A|E=0) = p(A,E=0)\,/\,p(E=0)$

Bucket Elimination



Algorithm *BE-bel* [Dechter 1996]

$$p(A|E=0) = \alpha \sum_{e,d,c,b} p(A)\, p(b|A)\, p(c|A)\, p(d|A,b)\, p(e|b,c)\, \mathbb{1}[e=0]$$

$$\sum_{b} \prod$$ ⟵ Elimination & combination operators

Time and space exponential in the induced-width / treewidth

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bucket A: $p(A)$ $\lambda_{E \to A}(A)$

induced width (max clique size)

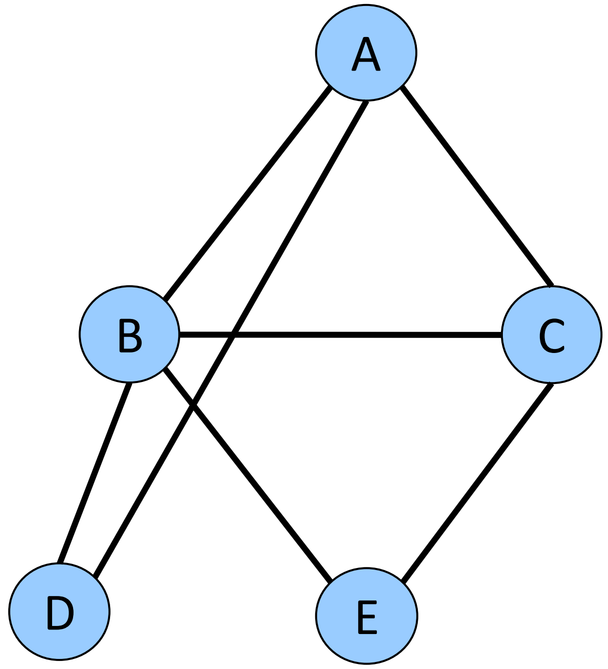
$p(E=0)$

$$p(A|E=0) = p(A, E=0) \,/\, p(E=0)$$

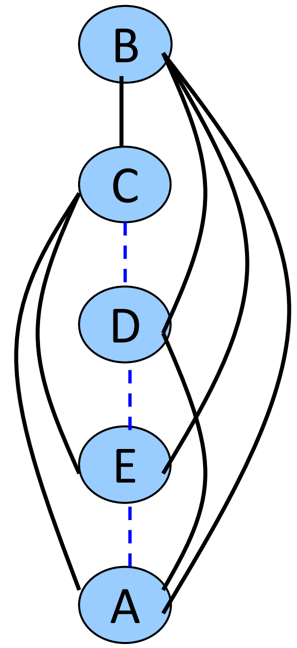
Induced Width (continued)

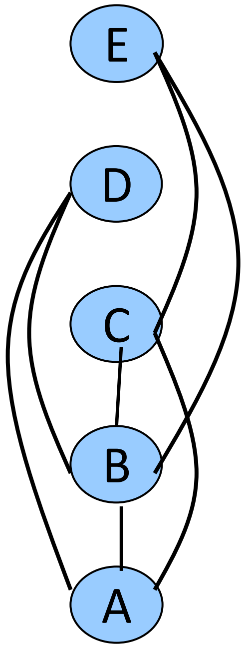
$w^*(d)$ – the induced width of the primal graph along ordering d

The effect of the ordering:



Primal (moraal) graph



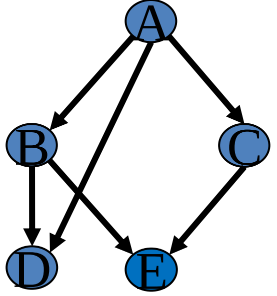


$w^*(d_1) = 4$

$w^*(d_2) = 2$

The impact of evidence?

Algorithm *BE-bel*



$$P(A \mid E=0) = \alpha \sum_{E=0,D,C,B} P(A) \cdot P(B \mid A) \cdot P(C \mid A) \cdot P(D \mid A,B) \cdot P(E \mid B,C)$$

$\sum_b \prod$ ← Elimination operator

bucket B: P(b|a) P(d|b,a) P(e|b,c)

bucket C: P(c|a) $\lambda^B(a,d,c,e)$

bucket D: $\lambda^C(a,d,e)$

bucket E: e=0 $\lambda^D(a,e)$

bucket A: P(a) $\lambda^E(a)$

P(e=0)

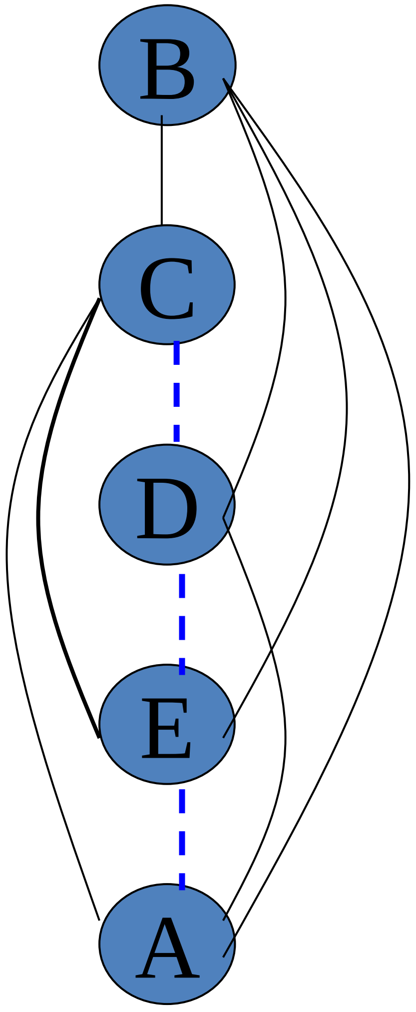
P(a|e=0)

W*=4

"induced width"

(max clique size)

B=1



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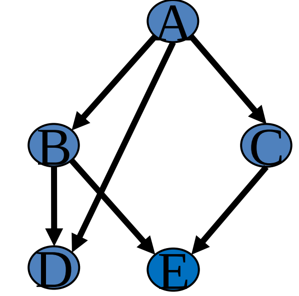
The impact of evidence?

Algorithm *BE-bel*



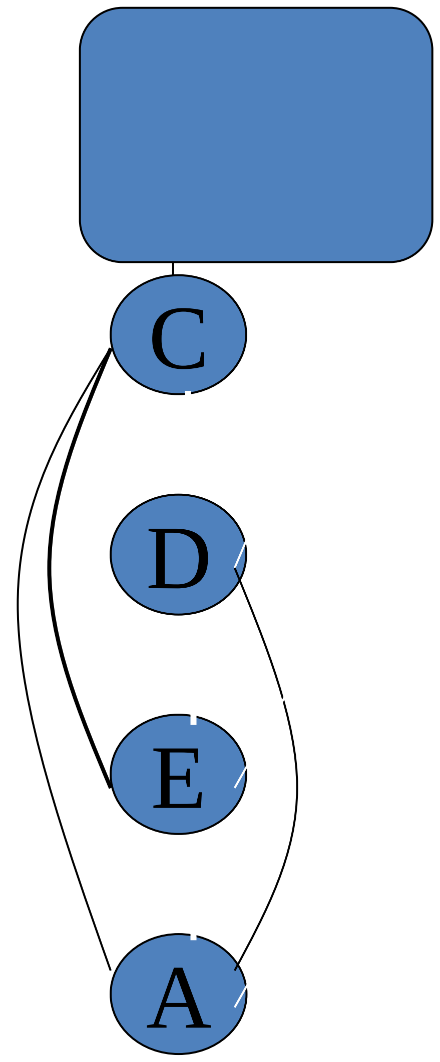
$$P(A \mid E=0) = \alpha \sum_{E=0,D,C,B} P(A) \cdot P(B \mid A) \cdot P(C \mid A) \cdot P(D \mid A,B) \cdot P(E \mid B,C)$$

P(A|E=0,B=1)?

$\sum_b \prod$ ← Elimination operator

bucket B: P(b|a) P(d|b,a) P(e|b,c)

B=1



bucket C: P(c|a) P(e|b=1,c)

bucket D: P(d|b=1,a)

bucket E: e=0

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bucket A: P(a) P(b=1|a)

P(e=0)

P(a|e=0)

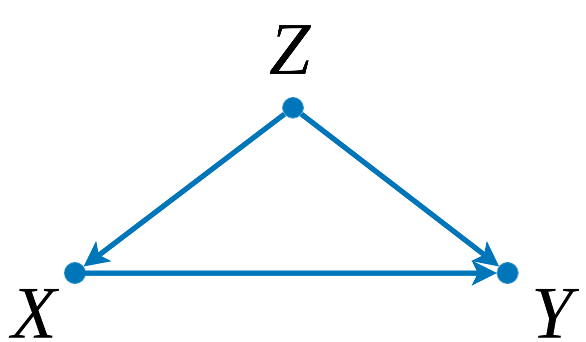
$$P(a \mid e=0) = \frac{P(a,e=0)}{P(e=0)}$$

Back to SCM

Markovian Case

- The distribution $P(\mathbf{v})$ decomposes as:

$$P(\mathbf{v}) = \sum_{\mathbf{u}} P(\mathbf{u}) \prod_{V_i \in \mathbf{V}} P(v_i \mid v_1, \ldots, v_{i-1}, \mathbf{u}) = \sum_{\mathbf{u}} P(\mathbf{u}) \prod_{V_i \in \mathbf{V}} P(v_i \mid pa_i, u_i)$$



$$P(z, x, y) = \sum_{\mathbf{u}} P(\mathbf{u}) P(z \mid u_z) P(x \mid z, u_x) P(y \mid x, z, u_y)$$

$$= \left(\sum_{u_z} P(z \mid u_z) P(u_z) \right) \left(\sum_{u_x} P(x \mid z, u_x) P(u_x) \right) \left(\sum_{u_y} P(y \mid x, z, u_y) P(u_y) \right)$$

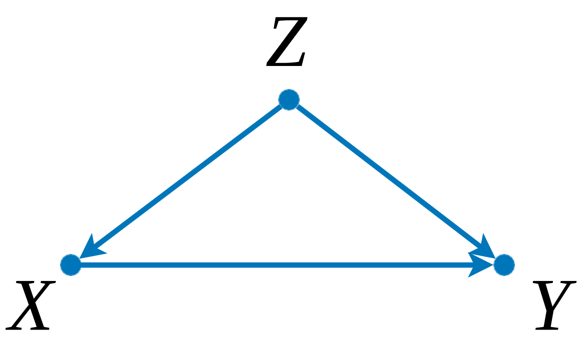
$$= P(z) P(x \mid z) P(y \mid x, z)$$

- In Markovian models, $P(v_i \mid pa_i)$ can be seen as “canonical factors”.

Markovian Case

- Every $P(v_i|pa_i)$ is computable from $P(\mathbf{v})$, *i.e.*,

$$P(v_i \mid pa_i) = \frac{\sum_{v \backslash v_i, pa_i} P(v)}{\sum_{v \backslash pa_i} P(v)}$$



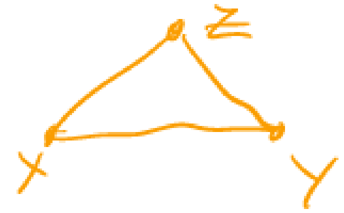
$$P(z, x, y) = P(z)P(x \mid z)P(y \mid x, z)$$

$$P(z) = \sum_{x,y} P(v)$$

$$P(y \mid x, z) = \frac{P(v)}{\sum_y P(v)}$$

$$P(x \mid z) = \frac{\sum_y P(v)}{\sum_{x,y} P(v)}$$

Markovian Case



U_Z: $P(U_Z)$, $P(Z|U_Z)$ →

U_X: $P(U_X)$, $P(X|U_X,Z)$

U_Y: $P(U_Y)$, $P(Y|Z,X,U_Y)$

X, Y, Z:

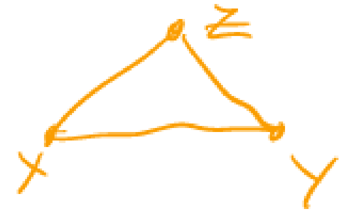
$$\lambda_{U_Z}(Z) = \sum_{U_Z} P(Z|U_Z)\cdot P(U_Z) = ?$$

$$\lambda_{U_X}(X,Z) = \sum_{U_X} P(X|U_X,Z)\cdot P(U_X) = ?$$

$$\lambda_{U_Y}(Y,X,Z) = \sum_{U_Y} P(Y|Z,X,U_Y)\cdot P(U_Y) = ?$$

$$P(X,Y,Z) = \lambda_{U_Z}(Z)\cdot\lambda_{U_X}(X,Z)\cdot\lambda_{U_Y}(Y,X,Z)$$

Markovian Case



U_Z: $P(U_Z)$, $P(Z|U_Z)$ →

U_X: $P(U_X)$, $P(X|U_X,Z)$

U_Y: $P(U_Y)$, $P(Y|Z,X,U_Y)$

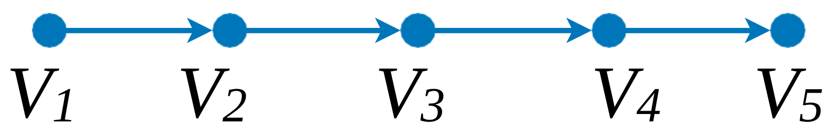
$$X,Y,Z: \begin{cases} \lambda_{U_Z}(Z) = \sum_{U_Z} P(Z|U_Z)\cdot P(U_Z) = ?\ P(Z) \\ \lambda_{U_X}(X,Z) = \sum_{U_X} P(X|U_X,Z)\cdot P(U_X) = ?\ P(X|Z) \\ \lambda_{U_Y}(Y,X,Z) = \sum_{U_Y} P(Y|Z,X,U_Y)\cdot P(U_Y) = ?\ P(Y|Z,X) \end{cases}$$

$$P(X,Y,Z) = \lambda_{U_Z}(Z)\cdot \lambda_{U_X}(X,Z)\cdot \lambda_{U_Y}(Y,X,Z)$$

P(X,Y,Z) = P(Z)P(X|Z)P(Y|X,Y,Z)

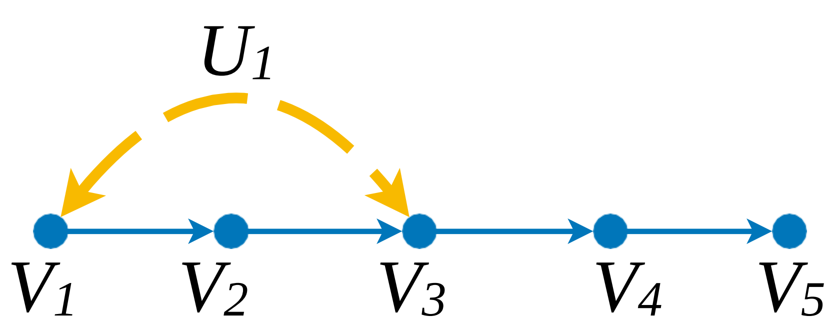
Semi-Markovian Case

- Start from a simple Markovian model:



$$P(\mathbf{v}) = P(v_1)P(v_2 \mid v_1)P(v_3 \mid v_2)P(v_4 \mid v_3)P(v_5 \mid v_4)$$

- Let's add an unobservable U_1, that affects two observables, and breaking Markovianity:



$$P(\mathbf{v}) = \sum_{u_1} P(u_1)P(v_1 \mid u_1)P(v_2 \mid v_1)P(v_3 \mid v_2, u_1)P(v_4 \mid v_3)P(v_5 \mid v_4)$$

$$= P(v_2 \mid v_1)P(v_4 \mid v_3)P(v_5 \mid v_4)\left(\sum_{u_1} P(u_1)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1) \right)$$

Using Bucket Elimination

U_1: $P(U_1)$, $P(V_1|U_1)$, $P(V_3|V_2,U_1)$

V_3: $P(V_4|V_3)$ $\lambda(V_1,V_3,V_2) = \sum_{U_1} P(V_3|V_2,U_1)\cdot P(V_1|U_1)\cdot P(U_1)$

V_2: $P(V_2|V_1)$

V_1:

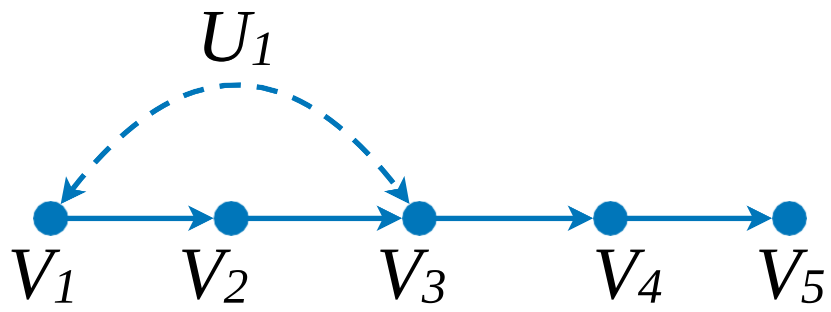
V_4: $P(V_5|V_4)$

V_5:

Can this be expressed using $P(V)$ only?

Semi-Markovian Case

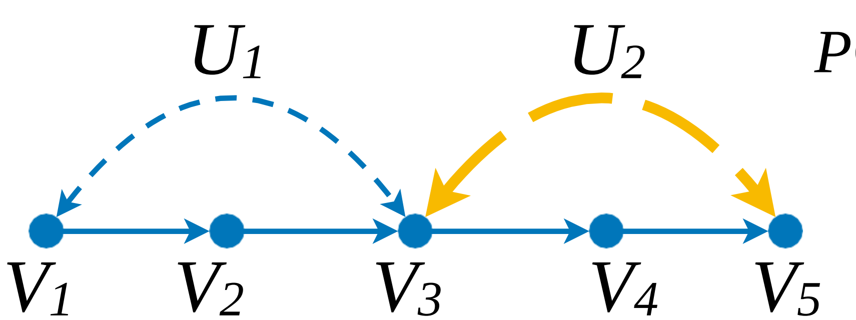
- From the previous model …



$$P(\mathbf{v}) = \sum_{u_1} P(u_1)P(v_1 \mid u_1)P(v_2 \mid v_1)P(v_3 \mid v_2, u_1)P(v_4 \mid v_3)P(v_5 \mid v_4)$$

$$= P(v_2 \mid v_1)P(v_4 \mid v_3)P(v_5 \mid v_4)\left(\sum_{u_1} P(u_1)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1) \right)$$

- Add another unobservable U_2,

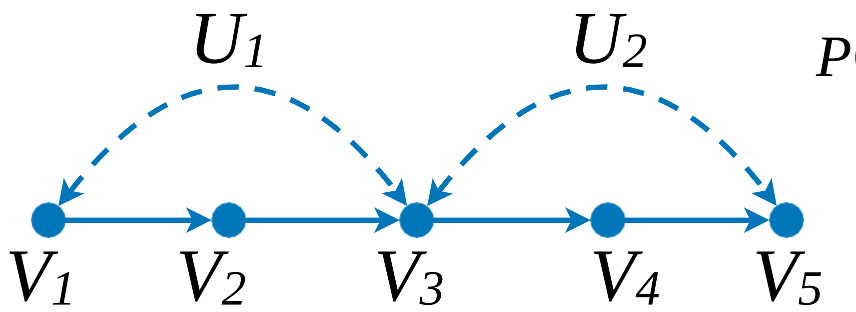


$$P(\mathbf{v}) = \sum_{u_1, u_2} P(u_1, u_2)P(v_1 \mid u_1)P(v_2 \mid v_1)P(v_3 \mid v_2, u_1, u_2)P(v_4 \mid v_3)P(v_5 \mid v_4, u_2)$$

$$= P(v_2 \mid v_1)P(v_4 \mid v_3)\left(\sum_{u_1, u_2} P(u_1, u_2)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1, u_2)P(v_5 \mid v_4, u_2) \right)$$

Semi-Markovian Case

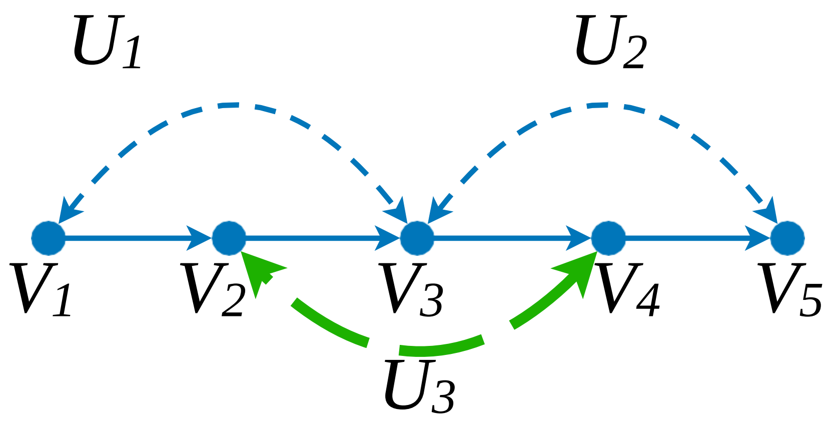
- From the previous model…



$$P(\mathbf{v}) = \sum_{u_1,u_2} P(u_1,u_2)P(v_1 \mid u_1)P(v_2 \mid v_1)P(v_3 \mid v_2,u_1,u_2)P(v_4 \mid v_3)P(v_5 \mid v_4,u_2)$$

$$= P(v_2 \mid v_1)P(v_4 \mid v_3)\left(\sum_{u_1,u_2} P(u_1,u_2)P(v_1 \mid u_1)P(v_3 \mid v_2,u_1,u_2)P(v_5 \mid v_4,u_2)\right)$$

- Let's add one more, U_3,

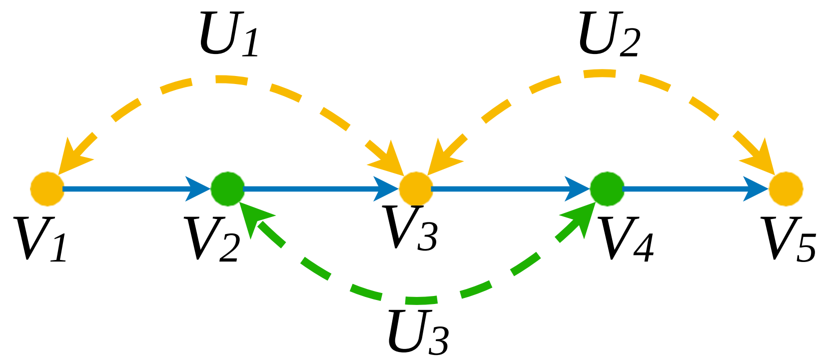


$$P(\mathbf{v}) = \sum_{u_1,u_2,u_3} P(u_1,u_2,u_3)P(v_1 \mid u_1)P(v_2 \mid v_1,u_3)P(v_3 \mid v_2,u_1,u_2)$$
$$P(v_4 \mid v_3,u_3)P(v_5 \mid v_4,u_2)$$

$$= \left(\sum_{u_3} P(u_3)P(v_2 \mid v_1,u_3)P(v_4 \mid v_3,u_3)\right)\left(\sum_{u_1,u_2} P(u_1,u_2)P(v_1 \mid u_1)P(v_3 \mid v_2,u_1,u_2)P(v_5 \mid v_4,u_2)\right)$$

C-factors

- Recall our example



$$P(\mathbf{v}) = \left(\sum_{u_3} P(u_3) P(v_2 \mid v_1, u_3) P(v_4 \mid v_3, u_3) \right) \left(\sum_{u_1, u_2} P(u_1, u_2) P(v_1 \mid u_1) P(v_3 \mid v_2, u_1, u_2) P(v_5 \mid v_4, u_2) \right)$$

- These factors made of sums may be long to write in terms of $P(\mathbf{v},\mathbf{u})$. However, their structure follows from the topology of the diagram, then we can abstract this concept out by defining a new function Q:

$$Q[\mathbf{C}](\mathbf{c}, pa_{\mathbf{c}}) = \sum_{u(\mathbf{C})} P(u(\mathbf{C})) \prod_{V_i \in \mathbf{C}} P(v_i \mid pa_i, u_i) \quad \text{where} \quad U(\mathbf{C}) = \bigcup_{V_i \in \mathbf{C}} U_i$$

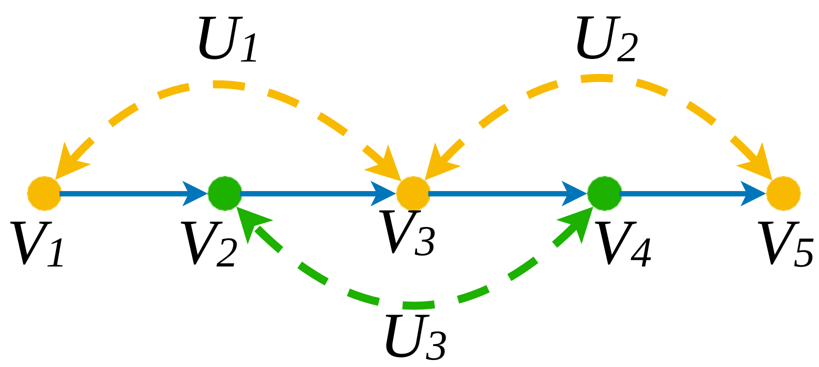
- Then $P(\mathbf{v})$ can be re-written as

$$P(\mathbf{v}) = Q[V_2, V_4](v_2, v_4, v_1, v_3) Q[V_1, V_3, V_5](v_1, v_3, v_5, v_2, v_4)$$

C-factors

- For convenience $Q[\mathbf{C}](\mathbf{c},pa_c)$ can be written just as $Q[\mathbf{C}]$
- Then, for our example, we can just write

$$P(\mathbf{v}) = Q[V_2, V_4]Q[V_1, V_3, V_5]$$



- No need to name the variables in $\boldsymbol{U}$ explicitly!
- Note that for the whole set of variables V

$$Q[\mathbf{V}] = \sum_{\mathbf{u}} P(\mathbf{u}) \prod_{V_i \in \mathbf{V}} P(v_i \mid pa_i, u_i) = P(\mathbf{v})$$

- For consistency define $Q[\varnothing]=1$

C-factors are Causal Effects

- Let $\boldsymbol{C} \subseteq \boldsymbol{V}$. Consider the causal effect of all other variables on $\boldsymbol{C}$, that is $P(\mathbf{c}|do(\mathbf{v}\backslash\mathbf{c}))$.
- By the truncated product we have

$$P(\mathbf{c} \mid do(\mathbf{v} \backslash \mathbf{c})) = \sum_{\mathbf{u}} \prod_{V_i \in \mathbf{C}} P(v_i \mid pa_i, u_i) P(\mathbf{u})$$

- All $\boldsymbol{U}$s that are not parents of any element in $\boldsymbol{C}$ can be summed out, hence

$$\boxed{P(\mathbf{c} \mid do(\mathbf{v} \backslash \mathbf{c}))} = \sum_{u(\mathbf{C})} P(u(\mathbf{C})) \prod_{V_i \in \mathbf{C}} P(v_i \mid pa_i, u_i) \boxed{= Q[C]}$$

This is a key connection between C-factors and causal effects.

On the completeness of an identifiability algorithm for semi-Ma models

Article *in* Annals of Mathematics and Artificial Intelligence · December 2008
DOI: 10.1007/s10472-008-9101-x · Source: DBLP

C-Factor (component)

Definition C-factor or C-component

A *c-component* (short for "confounded component," [3]) of variable set V on graph G consists of all the unobservable variables belonging to the same c-component related part of U and all observable variables that have an unobservable parent which is a member of that c-component.

Definition of Ancestral set

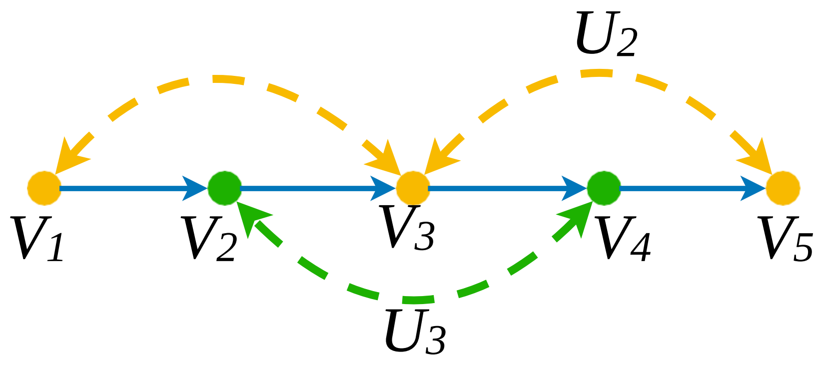
We conclude this section by giving several simple graphical definitions that will be needed later. For a given variable set $C \subseteq N$, let G_C denote the subgraph of G composed only of variables in C and all the bidirected links between variable pairs in C. We define $An(C)$ be the union of C and the set of observable ancestors of the variables in C in graph G and $De(C)$ be the union of C and the set of observable descendents of the variables in C in graph G.

An observable variable set $S \subseteq N$ in graph G is called an *ancestral set* if it contains all its own observed ancestors (i.e., $S = An(S)$).

Confounded components

27.13 Confounded Component

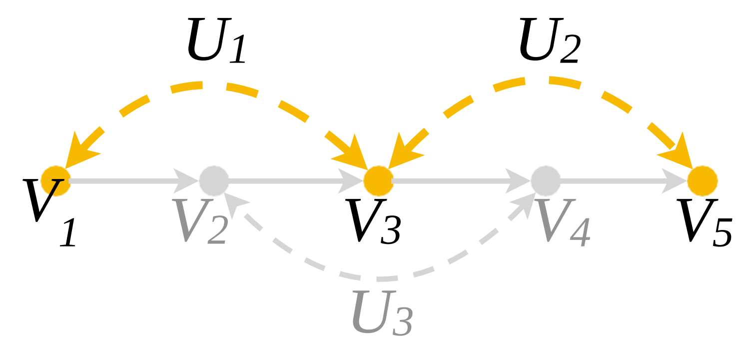
Let $\{\mathbf{C}_1, \mathbf{C}_2, \ldots \mathbf{C}_k\}$ be a partition over the set $\mathbf{V}$. $\mathbf{C}_i$ is said to be a confounded component (C-component) of $\mathcal{G}$ if there exists a path made of bidirected edges between V_i and V_j, for every $V_i, V_j \in \mathbf{C}_i$ in $\mathcal{G}$, and $\mathbf{C}_i$ is maximal.



Marginalizing Variables in C-factors

- To a certain extent, c-factors behave as its probabilistic counterparts.
- Consider the c-factor $Q[V_1,V_3,V_5]$ in our example

$$Q[V_1, V_3, V_5] = \sum_{u_1,u_2} P(u_1, u_2)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1, u_2)P(v_5 \mid v_4, u_2)$$



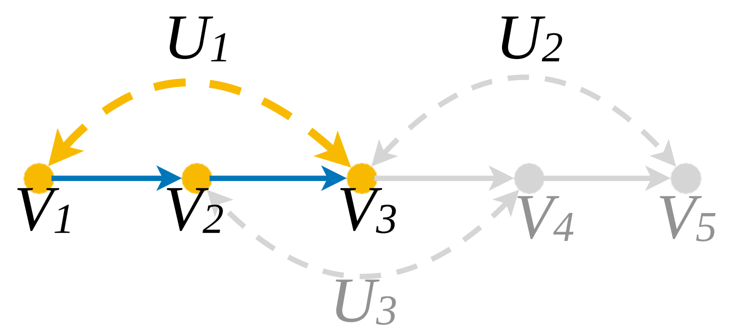
- Variables V_1,V_3,V_5 only appear in one term, because they are not the parent of any other variable in the factor. So, if we sum $Q[V_1,V_3,V_5]$ over any of them, for instance V_3, we have

$$\sum_{v_3} Q[V_1, V_3, V_5] = \sum_{v_3} \sum_{u_1,u_2} P(u_1, u_2)P(v_1, u_1)P(v_3 \mid v_2, u_1, u_2)P(v_5 \mid v_4, u_2)$$

$$= \sum_{u_1,u_2} P(u_1, u_2)P(v_1 \mid u_1)P(v_5 \mid v_4, u_2) = Q[V_1, V_5]$$

Marginalizing Variables in C-factors

- Consider now a different c-factor $Q[V_1,V_2,V_3]$,



By Definition of Q:

$$Q[V_1, V_2, V_3] = \sum_{u_1,u_2,u_3} P(u_1,u_2,u_3)P(v_1 \mid u_1)P(v_2 \mid v_1,u_3)P(v_3 \mid v_2,u_1,u_2)$$

- In contrast to the previous case, here V_1 appears in two terms since it's a parent of another variable in the factor. So, if we sum $Q[V_1,V_2,V_3]$ over V_1, we have

$$\sum_{v_1} Q[V_1,V_2,V_3] = \sum_{v_1}\sum_{u_1,u_2,u_3} P(u_1,u_2,u_3)P(v_1 \mid u_1)P(v_2 \mid v_1,u_3)P(v_3 \mid v_2,u_1,u_2)$$

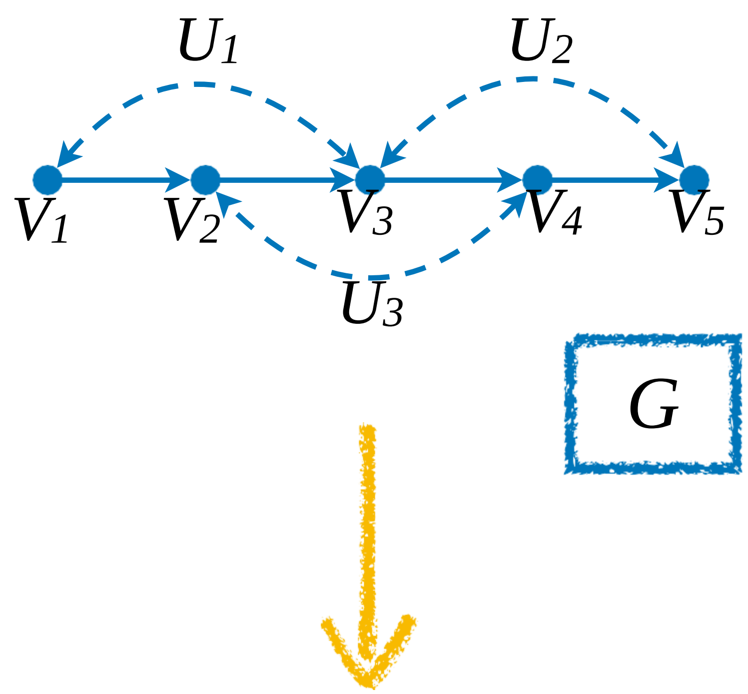
$$= \sum_{u_1,u_2} P(u_1,u_2)P(v_3 \mid v_2,u_1,u_2) \sum_{v_1,u_3} P(u_3)P(v_1 \mid u_1)P(v_2 \mid v_1,u_3) \neq Q[V_2,V_3]$$

Can we remove V1 here? Symbolically?

What variables can be marginalized in the logic of C-factors?

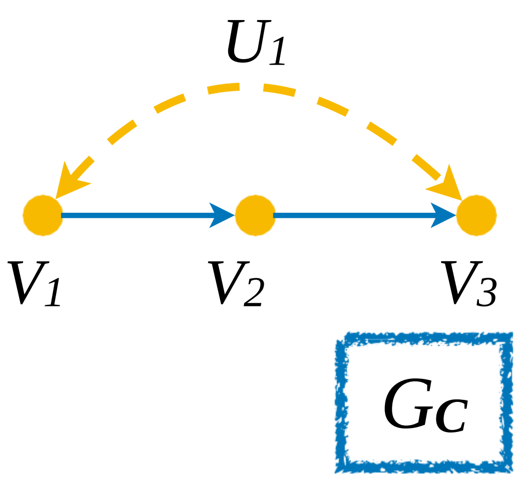
- Let $\boldsymbol{W} \subset \boldsymbol{C} \subseteq \boldsymbol{V}$, be two sets of variables.

- Lemma (ancestral-reduction). If $\boldsymbol{W}$ is ancestral, that is, it contains all $An(\boldsymbol{W})$ present in the subgraph made of the variables in $\boldsymbol{C}$, i.e., G_C.



- Then, $$Q[\mathbf{W}] = \sum_{\mathbf{c} \setminus \mathbf{w}} Q[\mathbf{C}]$$

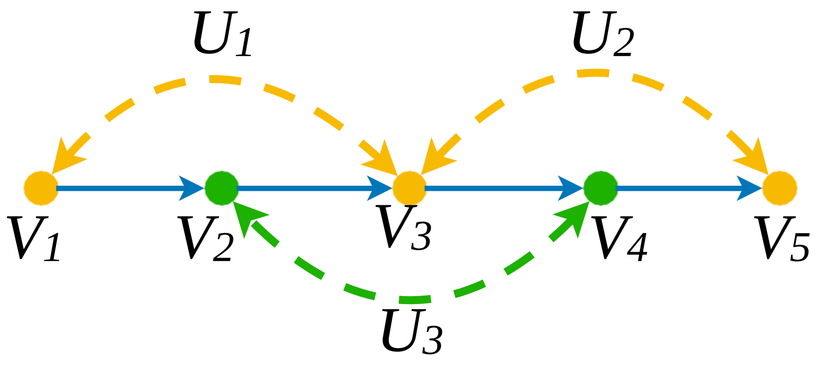
- For example, for $\boldsymbol{C}=\{V_1,V_2,V_3\}$, in G_C
  - $\boldsymbol{W}=\{V_1,V_2\}$ is ancestral $$Q[V_1, V_2] = \sum_{v_3} Q[V_1, V_2, V_3]$$
  - $\boldsymbol{W}=\{V_1\}$ is ancestral $$Q[V_1] = \sum_{v_2, v_3} Q[V_1, V_2, V_3]$$
  - $\boldsymbol{W}=\{V_2,V_3\}$ is not ancestral



So if Q(C) is identifiable then Q(W) is identifiable.

Confounded Components (C-Components)

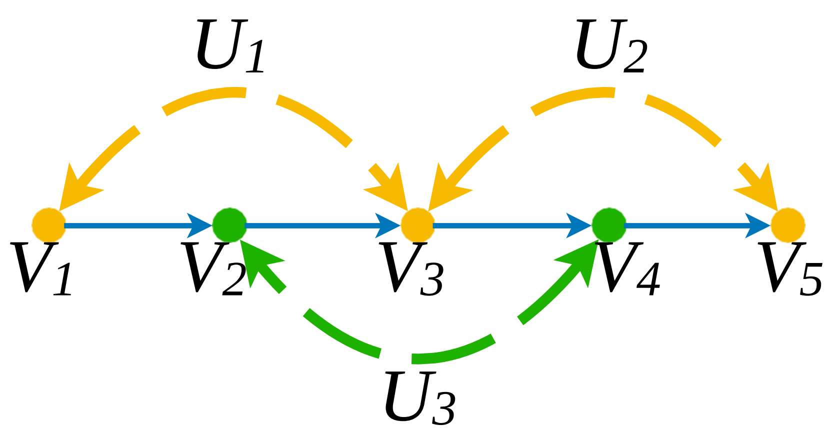
- Recall our example



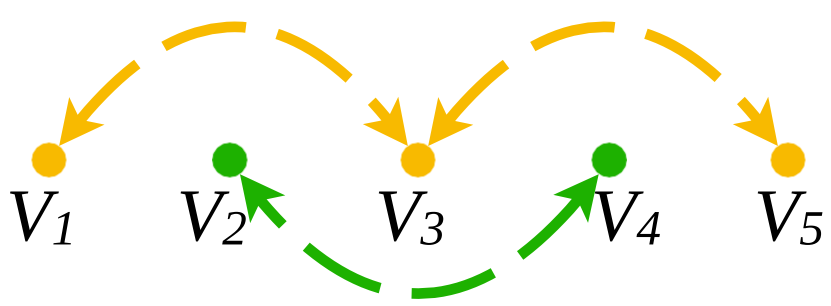
$$P(\mathbf{v}) = \left(\sum_{u_3} P(u_3)P(v_2 \mid v_1, u_3)P(v_4 \mid v_3, u_3) \right) \left(\sum_{u_1,u_2} P(u_1, u_2)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1, u_2)P(v_5 \mid v_4, u_2) \right)$$

- Definition (C-component). When V_i and V_j share a common unobservable parent U, $P(v_i \mid pa_i, u_i)$ and $P(v_j \mid pa_j, u_j)$ are tied together by the sum over $U \in U_i \cap U_j$. Then, we say that V_i and V_j are in the same confounded component (C-Component, for short).

C-Component Relationship



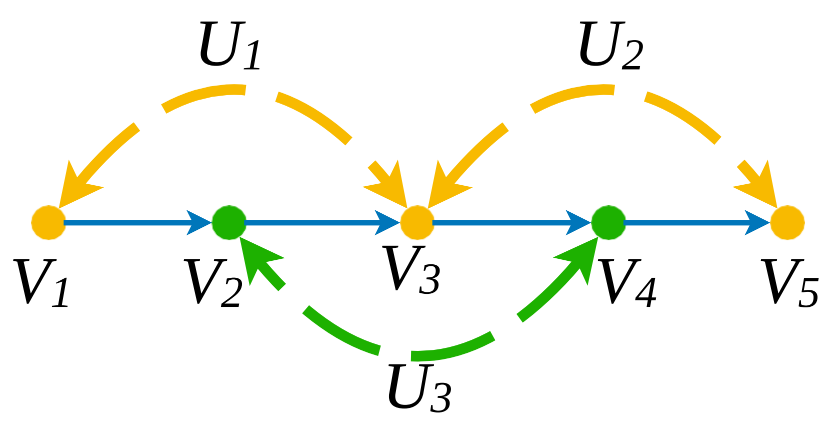
- V_1 is in the same c-component as V_3,
- V_3 is in the same c-component as V_5,
- By extension, V_1 is in the same c-component as V_5 too.
- V_2 is in the same c-component as V_4.



- To see it easily, consider the graph induced over the bidirected edges!
- Obs. The C-Component relation defines a partition over the observable variables, hence it is *Reflexive*, *Symmetric* and *Transitive*.

C-Component Factorization

- The distribution *P(**v**)* factorizes into c-factors associated with the c-components of the graph.



$$Q_1 = \{V_2, V_4\} \quad Q_2 = \{V_1, V_3, V_5\}$$

$$P(\mathbf{v}) = \left(\sum_{u_3} P(u_3)P(v_2 \mid v_1, u_3)P(v_4 \mid v_3, u_3)\right)\left(\sum_{u_1,u_2} P(u_1,u_2)P(v_1 \mid u_1)P(v_3 \mid v_2, u_1, u_2)P(v_5 \mid v_4, u_2)\right)$$

$$P(\mathbf{v}) = Q[V_2, V_4]\; Q[V_1, V_3, V_5]$$

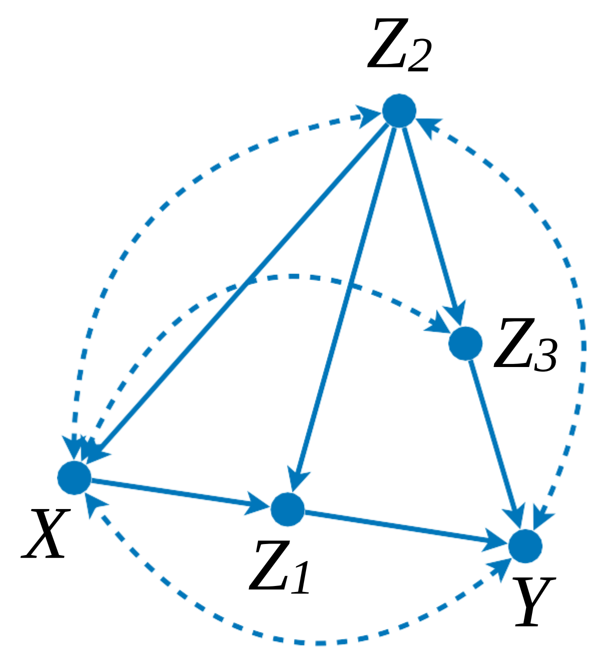
C-Component Factorization

- For any $\boldsymbol{H} \subseteq \boldsymbol{V}$, consider a graph $G_{\boldsymbol{H}}$.
- Let $H_1, H_2, \ldots, H_k$ be the c-components of $G_{\boldsymbol{H}}$.
- Then

$$Q[\mathbf{H}] = \prod_j Q[H_j]$$

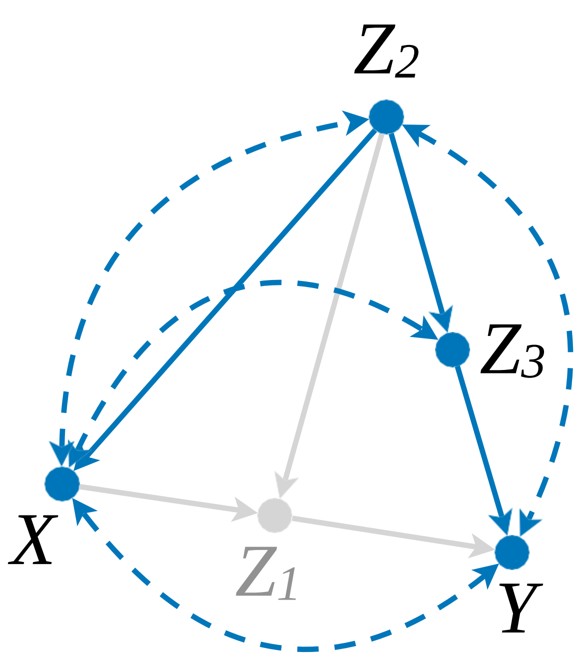
C-Component Factorization

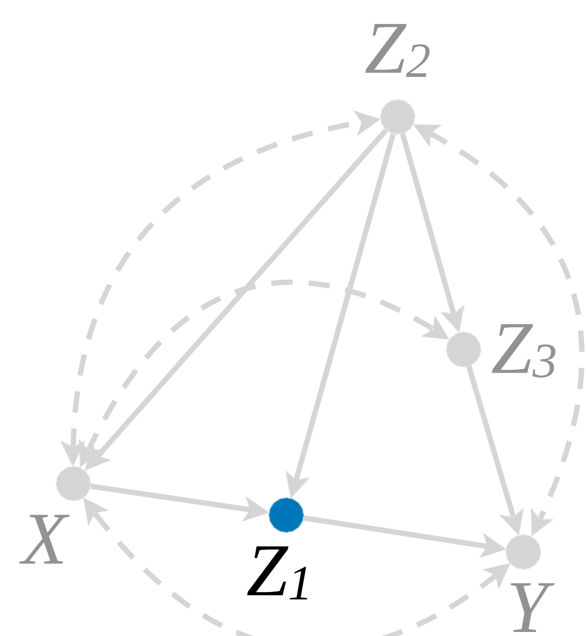
- Consider another example

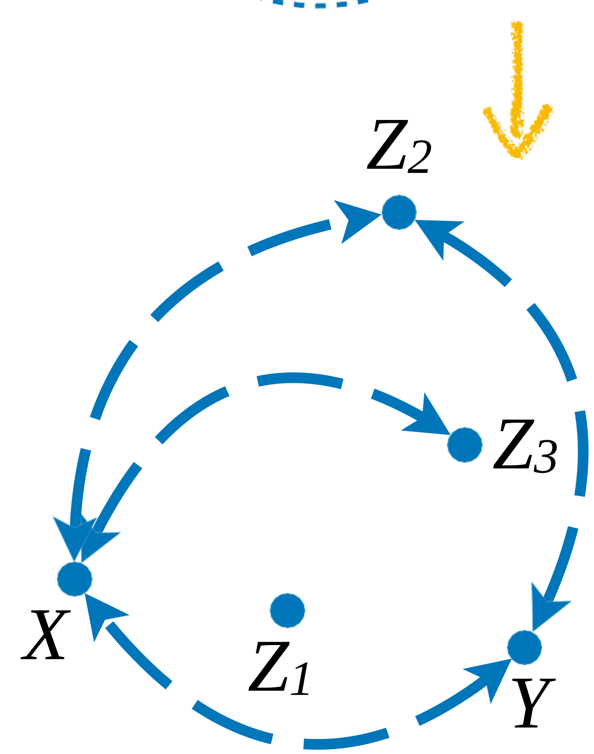


$$Q[\mathbf{V}] = Q[Z_2, Z_3, X, Y]Q[Z_1]$$

$$P(\mathbf{v}) = P(z_2, z_3, x, y \mid do(z_1))P(z_1 \mid do(z_2, z_3, x, y))$$







C-Component Factorization

- For any $\boldsymbol{H} \subseteq \boldsymbol{V}$, consider a graph $G_{\boldsymbol{H}}$.
- Let $H_1, H_2, \ldots, H_k$ be the c-components of $G_{\boldsymbol{H}}$.
- Then

$$Q[\mathbf{H}] = \prod_j Q[H_j]$$

C-Component Factorization

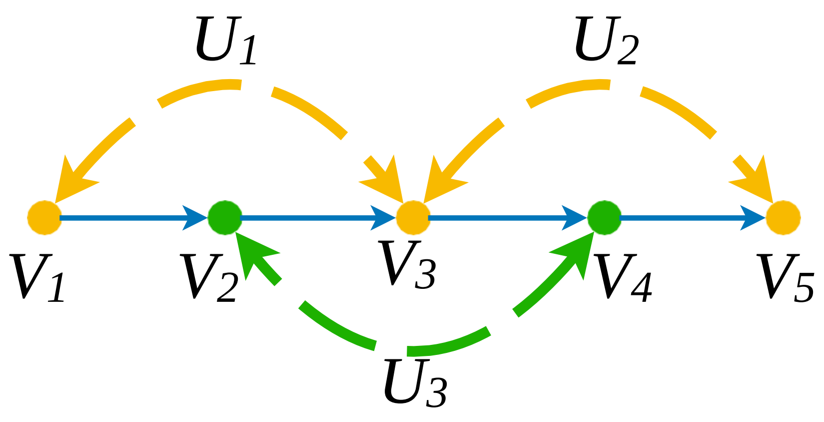
(Continued)

- Let $V_{h_1} < V_{h_2} < \ldots < V_{h_n}$ be a topological order over the variables in $\boldsymbol{H}$ according to G.
- Let $H^{\leq i}$ be the variables in $\boldsymbol{H}$ that come before V_{h_i}, including V_{h_i}.
- Let $H^{>i}$ be the variables in $\boldsymbol{H}$ that come after V_{h_i}.
- Then

$$Q[H_j] = \prod_{V_i \in H_j} \frac{Q[\mathbf{H}^{\leq i}]}{Q[\mathbf{H}^{\leq i-1}]} \qquad Q[\mathbf{H}^{\leq i}] = \sum_{h^{>i}} Q[\mathbf{H}]$$

C-Component Factorization

- Suppose $\boldsymbol{H}=\mathbf{V}=\{V_1,V_2,V_3,V_4,V_5\}$ is ancestral in G_C.



$$Q[\mathbf{V}] = Q[V_1, V_3, V_5]Q[V_2, V_4]$$

$$Q[V_1, V_3, V_5] = \frac{Q[V_1]}{Q[\emptyset]} \frac{Q[V_1, V_2, V_3]}{Q[V_1, V_2]} \frac{Q[V_1, V_2, V_3, V_4, V_5]}{Q[V_1, V_2, V_3, V_4]}$$

$$Q[V_2, V_4] = \frac{Q[V_1, V_2]}{Q[V_1]} \frac{Q[V_1, V_2, V_3, V_4]}{Q[V_1, V_2, V_3]}$$

$$Q[H_j] = \prod_{V_i \in H_j} \frac{Q[\mathbf{H}^{\leq i}]}{Q[\mathbf{H}^{\leq i-1}]}$$

C-Component Factorization

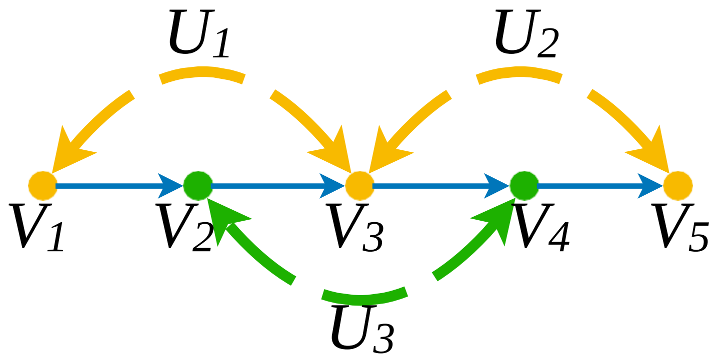
$$Q[V_1, V_3, V_5] = \frac{Q[V_1]}{Q[\varnothing]} \frac{Q[V_1, V_2, V_3]}{Q[V_1, V_2]} \frac{Q[V_1, V_2, V_3, V_4, V_5]}{Q[V_1, V_2, V_3, V_4]}$$

$$= \frac{\sum_{v_2,v_3,v_4,v_5} Q[\mathbf{V}]}{\sum_{v_1,v_2,v_3,v_4,v_5} Q[\mathbf{V}]} \frac{\sum_{v_4,v_5} Q[\mathbf{V}]}{\sum_{v_3,v_4,v_5} Q[\mathbf{V}]} \frac{\sum_{\varnothing} Q[\mathbf{V}]}{\sum_{v_5} Q[\mathbf{V}]}$$

$$= \frac{\sum_{v_2,v_3,v_4,v_5} P(\mathbf{v})}{\sum_{v_1,v_2,v_3,v_4,v_5} P(\mathbf{v})} \frac{\sum_{v_4,v_5} P(\mathbf{v})}{\sum_{v_3,v_4,v_5} P(\mathbf{v})} \frac{\sum_{\varnothing} P(\mathbf{v})}{\sum_{v_5} P(\mathbf{v})}$$

$$Q[\mathbf{H}^{\leq i}] = \sum_{h^{>i}} Q[\mathbf{H}]$$

$$= \frac{P(v_1)}{1} \frac{P(v_1, v_2, v_3)}{P(v_1, v_2)} \frac{P(v_1, v_2, v_3, v_4, v_5)}{P(v_1, v_2, v_3, v_4)}$$



$$= P(v_1)P(v_3 \mid v_1, v_2)P(v_5 \mid v_1, v_2, v_3, v_4)$$

C-Component Factorization

$$Q[V_2, V_4] = \frac{Q[V_1, V_2]}{Q[V_1]} \frac{Q[V_1, V_2, V_3, V_4]}{Q[V_1, V_2, V_3]}$$

$$= \frac{\sum_{v_3,v_4,v_5} Q[\mathbf{V}]}{\sum_{v_2,v_3,v_4,v_5} Q[\mathbf{V}]} \frac{\sum_{v_5} Q[\mathbf{V}]}{\sum_{v_4,v_5} Q[\mathbf{V}]}$$

$$= \frac{\sum_{v_3,v_4,v_5} P(\mathbf{v})}{\sum_{v_2,v_3,v_4,v_5} P(\mathbf{v})} \frac{\sum_{v_5} P(\mathbf{v})}{\sum_{v_4,v_5} P(\mathbf{v})}$$

$$= \frac{P(v_1, v_2)}{P(v_1)} \frac{P(v_1, v_2, v_3, v_4)}{P(v_1, v_2, v_3)}$$

$$= P(v_2 \mid v_1) P(v_4 \mid v_1, v_2, v_3)$$

How to get just $Q[V_2]$ or $Q[V_4]$?

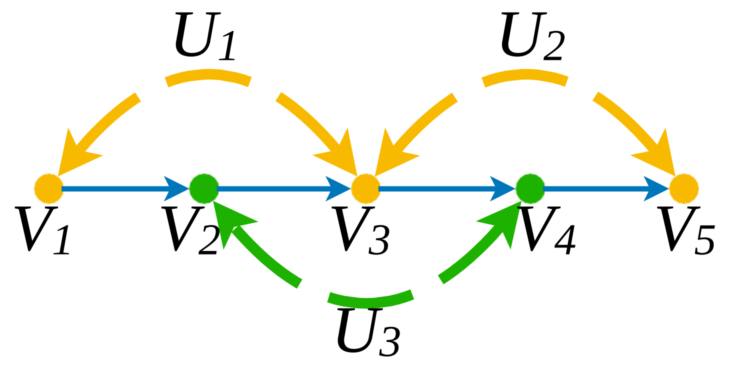
Both are ancestral in $G_{\{V_2,V_4\}}$!

$$Q[V_2] = \sum_{v_4} Q[V_2, V_4]$$

$$= P(v_2 \mid v_1)$$

$$Q[V_4] = \sum_{v_2} Q[V_2, V_4]$$

$$= \sum_{v_2} P(v_2 \mid v_1) P(v_4 \mid v_1, v_2, v_3)$$



C-Component Factorization

$$Q[V_2, V_4] = \frac{Q[V_1, V_2]}{Q[V_1]} \frac{Q[V_1, V_2, V_3, V_4]}{Q[V_1, V_2, V_3]}$$

$$= \frac{\sum_{v_3,v_4,v_5} Q[\mathbf{V}]}{\sum_{v_2,v_3,v_4,v_5} Q[\mathbf{V}]} \frac{\sum_{v_5} Q[\mathbf{V}]}{\sum_{v_4,v_5} Q[\mathbf{V}]}$$

$$= \frac{\sum_{v_3,v_4,v_5} P(\ldots}{\sum_{v_2,v_3,v_4,v_5} P\ldots}$$

$$= \frac{P(v_1, v_2)}{P(v_1)} \frac{P(\ldots)}{P(v_1, v_2, v_3)}$$

$$= P(v_2 \mid v_1) P(v_4 \mid v_1, v_2, v_3)$$

How to get just *Q[V₂]* or *Q[V₄]*?

Both are ancestral in $G_{\{V_2,V_4\}}$!

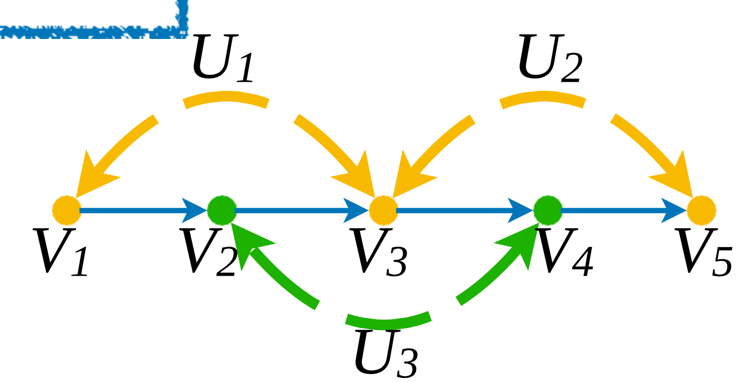
$$Q[V_2] = \sum_{v_4} Q[V_2, V_4]$$

$$= P(v_2 \mid v_1)$$

$$\ldots Q[V_2, V_4]$$

$$\ldots \mid v_1) P(v_4 \mid v_1, v_2, v_3)$$

Notice that these c-factors are expressible in terms of the obs. distribution (no ***U***-terms)!



C-factor Algebra - Summary

We have two basic operations over c-factors

1. Reduce to an ancestral set

$$Q[\mathbf{W}] = \sum_{\mathbf{c} \setminus \mathbf{w}} Q[\mathbf{C}]$$

If $\boldsymbol{W}$ is ancestral in $G_{\boldsymbol{C}}$

2. Factorize into c-components

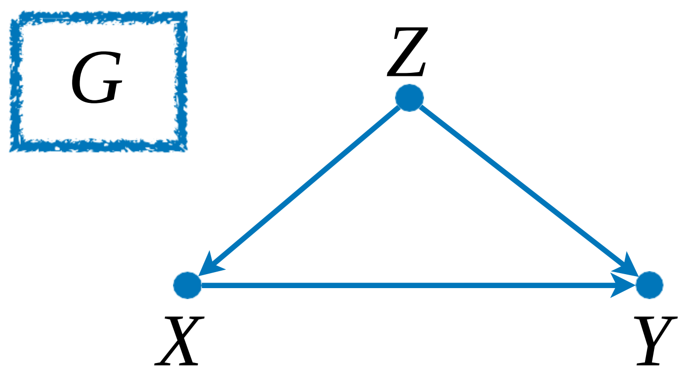
$$Q[\mathbf{H}] = \prod_{j} Q[H_j]$$

Where $H_1, \ldots H_k$, are the c-components in $G_{\boldsymbol{H}}$

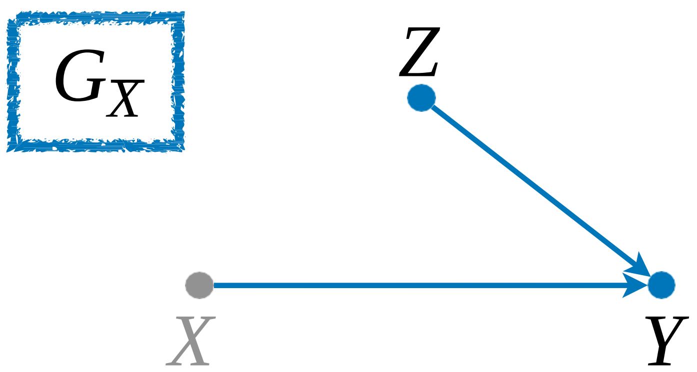
Expressing Causal Queries in terms of C-factors

Causal Effect in terms of C-Factors

- Consider an intervention $do(x)$



$$P(\mathbf{v}) = Q[Z]Q[X]Q[Y]$$



$$P(y \mid do(x)) = \sum_z P(y, z \mid do(x))$$
$$= \sum_z Q[Y, Z]$$
$$= \sum_z Q[Y]Q[Z]$$

Back-door!

$$P(y \mid do(x)) = \sum_z P(y \mid z, x)P(z)$$

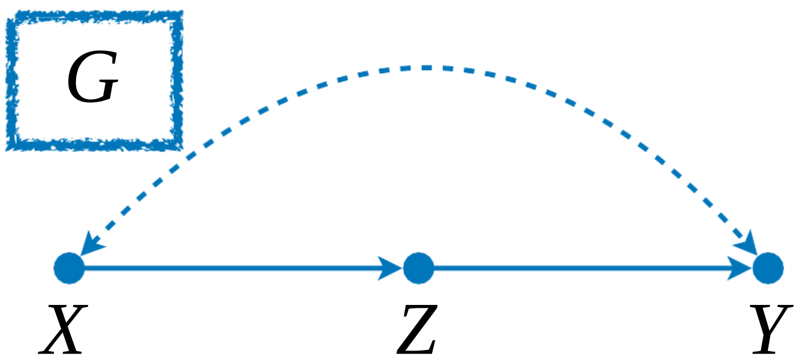
- We can get both $Q[Z]$ and $Q[Y]$ from $Q[\mathbf{V}]$ using c-component decomposition with G and $P(\mathbf{v})$.

$$Q[Z] = \frac{Q[Z]}{Q[\varnothing]} = \frac{\sum_{y,x} Q[Z, X, Y]}{\sum_{z,y,x} Q[Z, X, Y]} = P(z)$$

$$Q[Y] = \frac{Q[Z, X, Y]}{Q[Z, X]} = P(y \mid z, x)$$

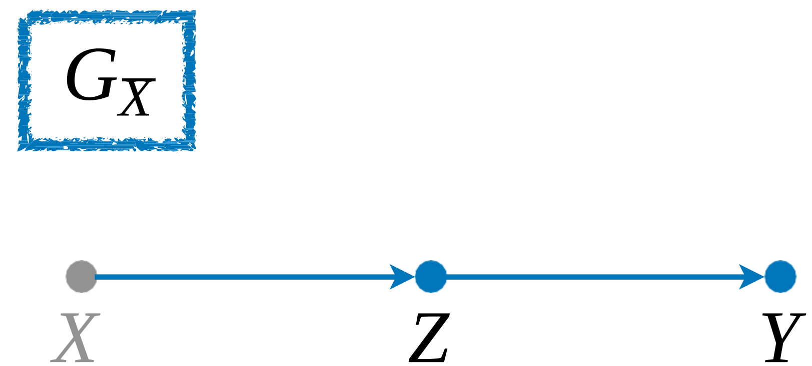
Causal Effect in terms of C-Factors

- Consider an intervention



$$P(\mathbf{v}) = Q[X, Y]\,Q[Z]$$

$$Q[Z] = \frac{Q[X, Z]}{Q[X]} = P(z \mid x)$$



$$P(y \mid do(x)) = \sum_z P(y, z \mid do(x))$$

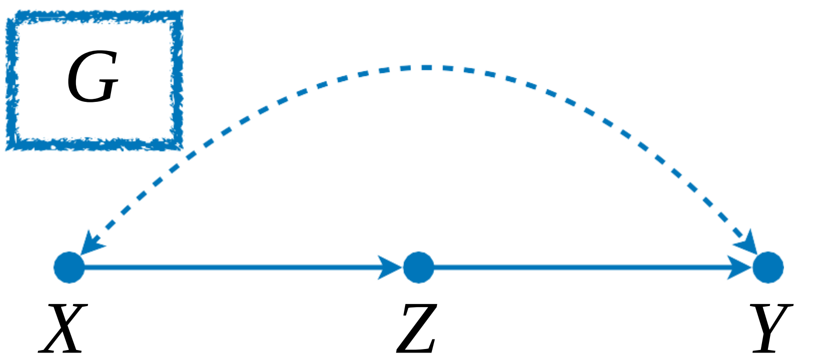
$$= \sum_z Q[Y, Z]$$

$$= \sum_z Q[Y]\,Q[Z]$$

- *Q[Z]* is the same in both
- Can we get *Q[Y]* from *Q[X,Y]* ?

Causal Effect in terms of C-Factors

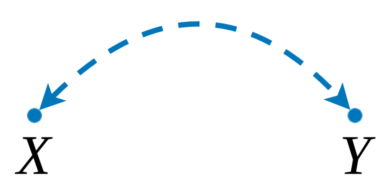
- Consider an intervention



$$Q[X,Y] = \frac{Q[X]}{Q[\varnothing]} \frac{Q[X,Z,Y]}{Q[X,Z]}$$

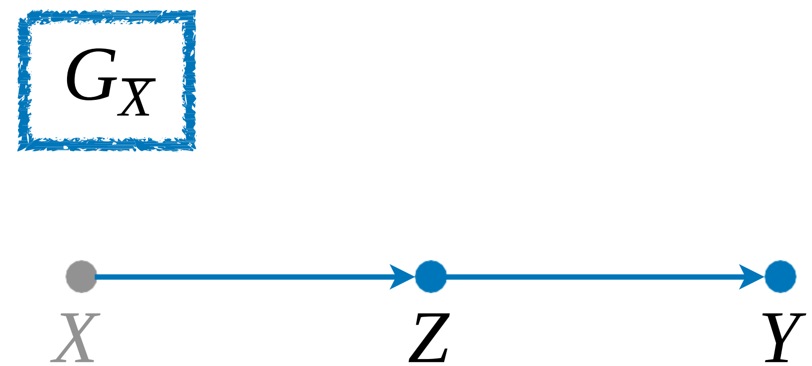
$$= P(x)P(y \mid x,z)$$

- $\{Y\}$ is ancestral in $G_{\{X,Y\}}$



$$Q[Y] = \sum_x Q[X,Y]$$

$$= \sum_x P(x)P(y \mid x,z)$$



$$P(y \mid do(x)) = \sum_z Q[Y]Q[Z]$$

$$= \sum_z \left(\sum_{x'} P(x')P(y \mid x',z) \right) P(z \mid x)$$

$$= \sum_z P(z \mid x) \sum_{x'} P(x')P(y \mid x',z)$$

Front-door!

A General Approach

A General Identification Algorithm

- Given G and the query variables $\boldsymbol{X},\boldsymbol{Y}$

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{v}\setminus(\mathbf{x}\cup\mathbf{y})} Q[\mathbf{V}\setminus\mathbf{X}]$$

$$= \sum_{\mathbf{d}\setminus\mathbf{y}} Q[\mathbf{D}]$$ where $\boldsymbol{D}=An(\boldsymbol{Y})$ in $G_{\overline{\mathbf{X}}}$

- Suppose the graph $G_{\boldsymbol{D}}$ has C-components $\mathbf{D}_1, \mathbf{D}_2, \ldots, \mathbf{D}_k$, then

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{\mathbf{d}\setminus\mathbf{y}} \prod_{i} Q[\mathbf{D}_i]$$

A General Identification Algorithm

Function Identify(C, T, Q)
INPUT: $C \subseteq T \subseteq V$, $Q = Q[T]$. Assuming G_T is composed of one single c-component.
OUTPUT: Expression for $Q[C]$ in terms of Q or fail to determine.

Let $A = An(C)_{G_T}$.

- IF $A = C$, output $Q[C] = \sum_{T \setminus C} Q$.
- IF $A = T$, output FAIL.
- IF $C \subset A \subset T$
  1. Assume that in G_A, C is contained in a c-component T'.
  2. Compute $Q[T']$ from $Q[A] = \sum_{T \setminus A} Q$ by Lemma 11.
  3. Output Identify$(C, T', Q[T'])$.

Figure 5.9: A function determining if $Q[C]$ is computable from $Q[T]$.

Completeness

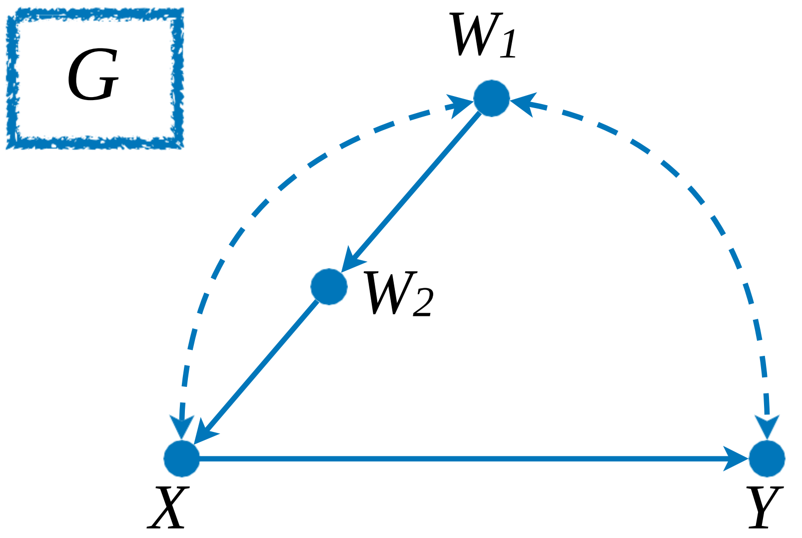
Theorem [Huang and Valtorta, 2008]

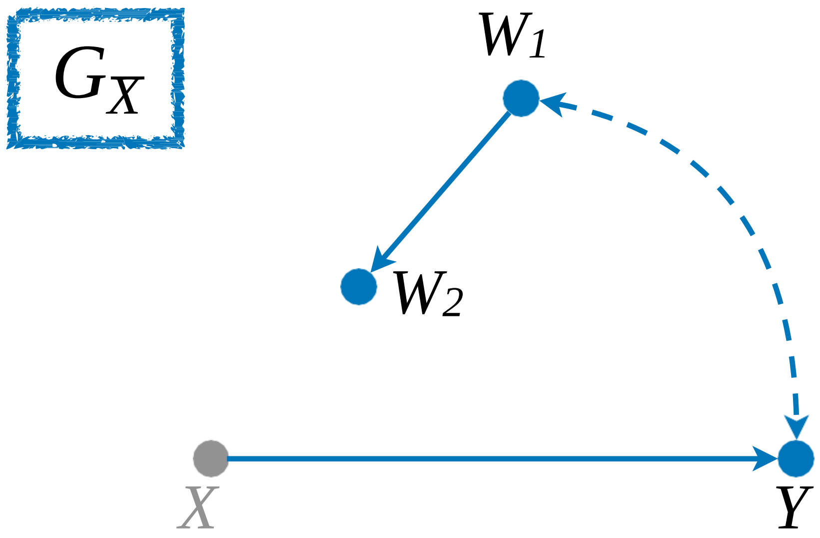
The causal effect $P(\mathbf{y}|do(\mathbf{x}))$ is identifiable from causal diagram G and $P(\mathbf{v})$ *if and only if* each of the C-factors $Q[\mathbf{D_i}$ is identifiable by `Identify(Di,Ci,Q[Ci],G)`.

Where $\mathbf{C}_i$ is the C-component of G containing $\mathbf{D}_i$.

Solving the Napkin

- Recall the Napkin graph from last time





$$P(\mathbf{v}) = Q[W_1, X, Y]Q[W_2]$$

$$P(y \mid do(x)) = \sum_{w_1, w_2} Q[W_1, W_2, Y]$$

- $Q[W_1,X,Y]$ is computable from $Q[\mathbf{V}]$
- Can we get $Q[Y]$ from $Q[W_1,X,Y]$?

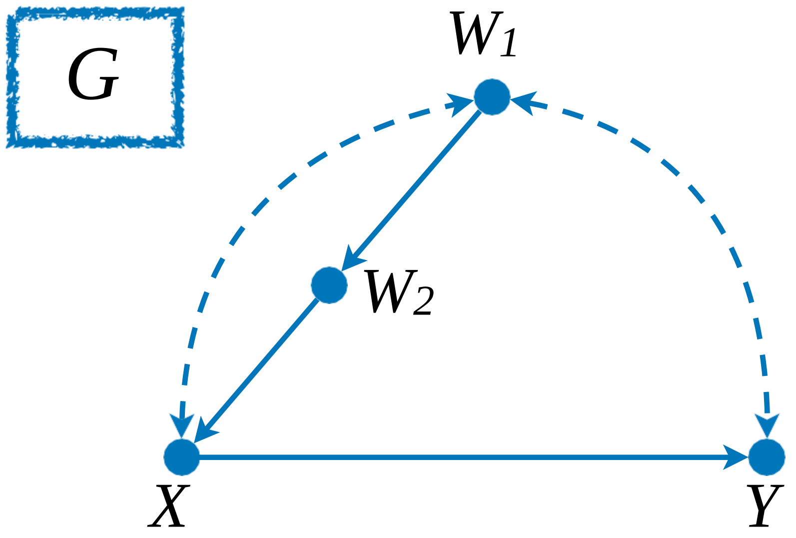
- $\{Y\}$ is ancestral in $G_{\{W_1,W_2,Y\}}$, so

$$P(y \mid do(x)) = Q[Y]$$

Solving the Napkin

- We can compute $Q[W_1,X,Y]$ from $Q[W_1,W_2,X,Y]$

$$P(y \mid do(x)) = Q[Y]$$



$$Q[W_1, X, Y] = \frac{Q[W_1]}{Q[\varnothing]} \frac{Q[W_1, W_2, X]}{Q[W_1, W_2]} \frac{Q[W_1, W_2, X, Y]}{Q[W_1, W_2, X]}$$

$$= \frac{P(w_1)}{1} \frac{P(w_1, w_2, x)}{P(w_1, w_2)} \frac{P(w_1, w_2, x, y)}{P(w_1, w_2, x)}$$

$$= P(w_1) P(x \mid w_1, w_2) P(y \mid w_1, w_2, x)$$

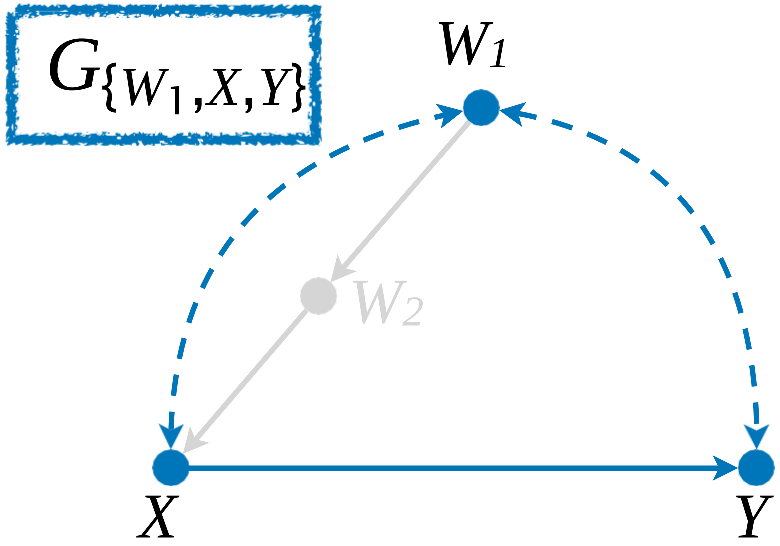
Solving the Napkin

- $\{X,Y\}$ is ancestral in $G_{\{W_1,X,Y\}}$

$$P(y \mid do(x)) = Q[Y]$$

$$Q[X,Y] = \sum_{w_1} Q[W_1, X, Y]$$

$$= \sum_{w_1} P(w_1)P(x \mid w_1, w_2)P(y \mid w_1, w_2, x)$$



Solving the Napkin

- $G_{\{X,Y\}}$ has two c-components, hence

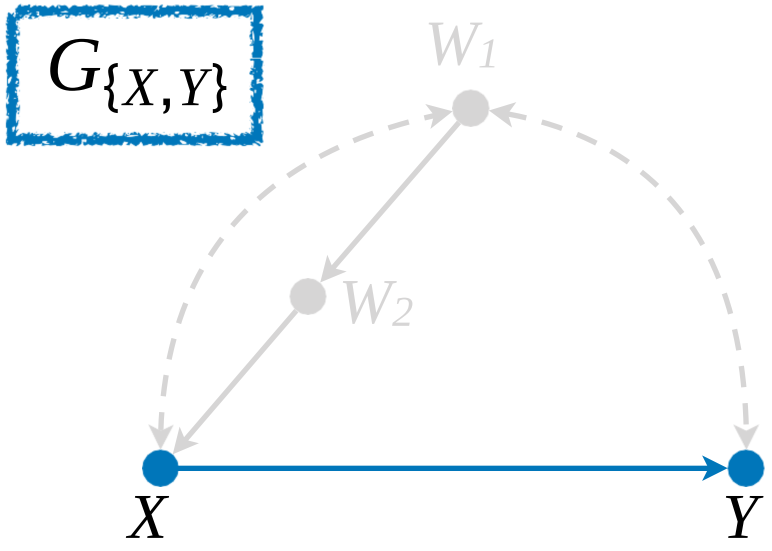
$$P(y \mid do(x)) = Q[Y]$$

$$Q[X,Y] = Q[X]Q[Y]$$

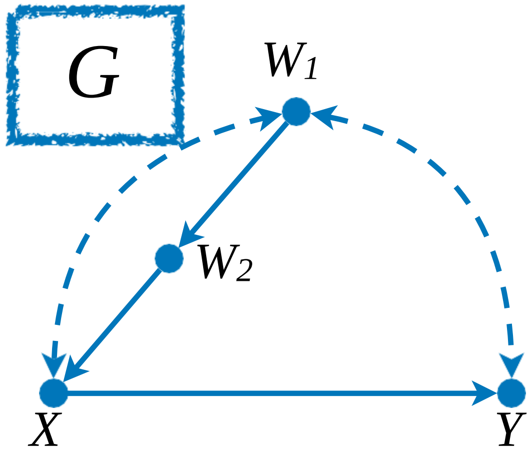
$$Q[Y] = \frac{Q[X,Y]}{Q[X]} = \frac{Q[X,Y]}{\sum_y Q[X,Y]}$$

$$= \frac{\sum_{w_1} P(w_1)P(x \mid w_1, w_2)P(y \mid w_1, w_2, x)}{\sum_{y,w_1} P(w_1)P(x \mid w_1, w_2)P(y \mid w_1, w_2, x)}$$

$$= \frac{\sum_{w_1} P(w_1)P(x \mid w_1, w_2)P(y \mid w_1, w_2, x)}{\sum_{w_1} P(w_1)P(x \mid w_1, w_2)} = P(y \mid do(x))$$



Solve the Napkin using Do-Calculus



- Let's see an equivalent do-calculus derivation

$$P(y \mid do(x)) = P(y \mid do(x, w_2, w_1))$$ Rule 3: $(Y \perp\!\!\!\perp W_2, W_1 \mid X)_{G_{\overline{X W_1, W_2}}}$

$$= P(y \mid x, do(w_2, w_1))$$ Rule 2: $(Y \perp\!\!\!\perp X \mid W_1, W_2)_{G_{\underline{X}\overline{W_1, W_2}}}$

$$= \frac{P(y, x \mid do(w_2, w_1))}{P(x \mid do(w_2, w_1))}$$ Conditional probability

$$= \frac{P(y, x \mid do(w_2))}{P(x \mid do(w_2))}$$ Rule 3: $(Y, X \perp\!\!\!\perp W_1 \mid W_2)_{G_{\overline{W_1}\overline{W_2}}}$

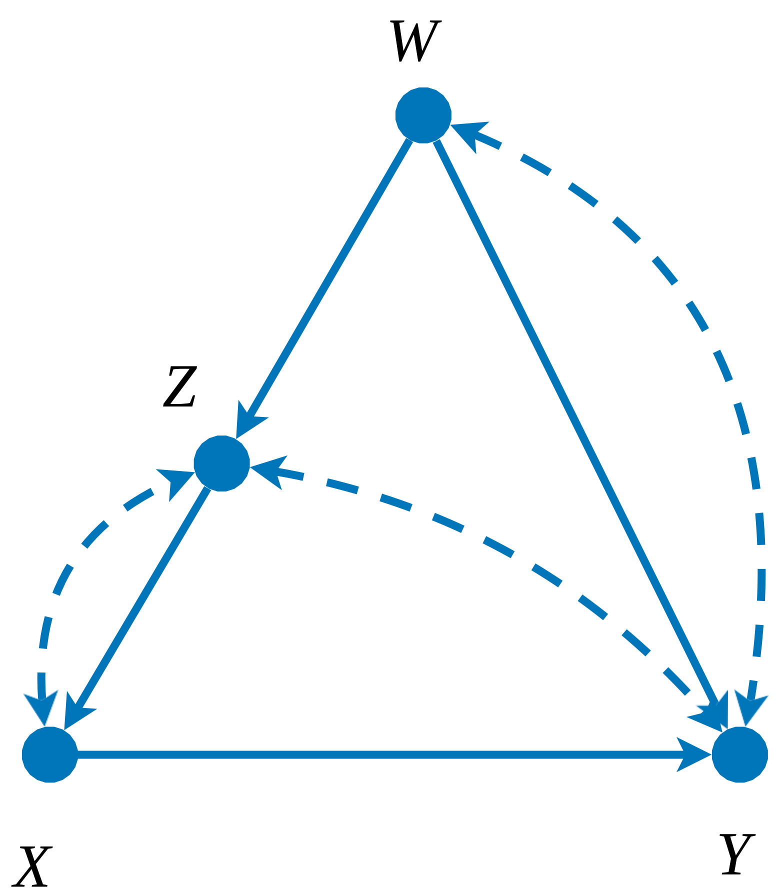
$$= \frac{\sum_{w_1} P(y, x \mid do(w_2), w_1) P(w_1 \mid do(w_2))}{\sum_{w_1} P(x \mid do(w_2), w_1) P(w_1 \mid do(w_2))}$$ Condition on W_1

$$= \frac{\sum_{w_1} P(y, x \mid w_2, w_1) P(w_1)}{\sum_{w_1} P(x \mid w_2, w_1) P(w_1)}$$ Rule 2: $(Y, X \perp\!\!\!\perp W_2 \mid W_1)_{G_{\underline{W_2}}}$

Rule 3: $(W_1 \perp\!\!\!\perp W_2)_{G_{\overline{W_2}}}$

Food for Thought

- Use the strategy discussed in this lecture to identify the effect $P(y|do(x))$ in the following causal diagram



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Project Inormation

Link

https://www.dropbox.com/scl/fi/985rnnmlzfllya2rxjccx/project-2024.docx?rlkey=tddqaa442otsf02xo2lhzecjk&dl=0