

Causal and Probabilistic Reasoning

Rina Dechter

Identification of Causal Effect Adjusting for parents

Primer chapter 3, Causality 1.3, 3.1, 3.2

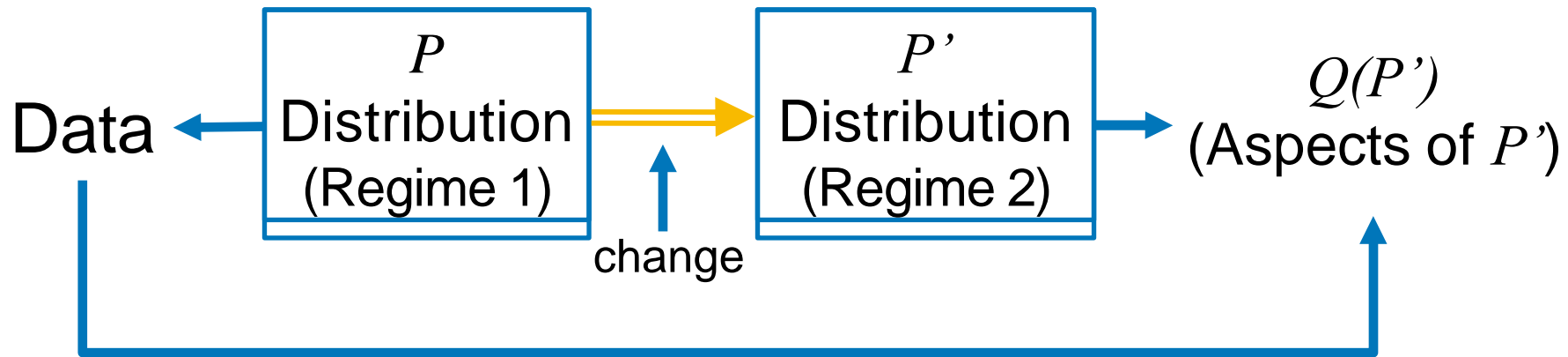
Outline (chapter 3)

- The semantic of Intervention in SCM, the do operators
- How to determine $P(Y | do(x))$ given an SCM
- The back door criterion and the adjustment formula
- Identifiability

Target: to Determine the Effect of Interventions

- “Correlation is no causation”, e.g., Increasing ice-cream sales is correlated with more crime, still selling more ice-cream will not cause more violence. Hot weather is a cause for both.
- **Randomized controlled experiments** are used to determine causation: all factors except a selected one of interest are kept static or random. So the outcome can only be influenced by the selected factor.
- Randomized experiments are often not feasible (we cannot randomize the weather), so how can we determine cause for wildfire?
- **Observational studies** must be used. But how we untangle correlation from causation?

Causal Inference — Connecting Different Worlds



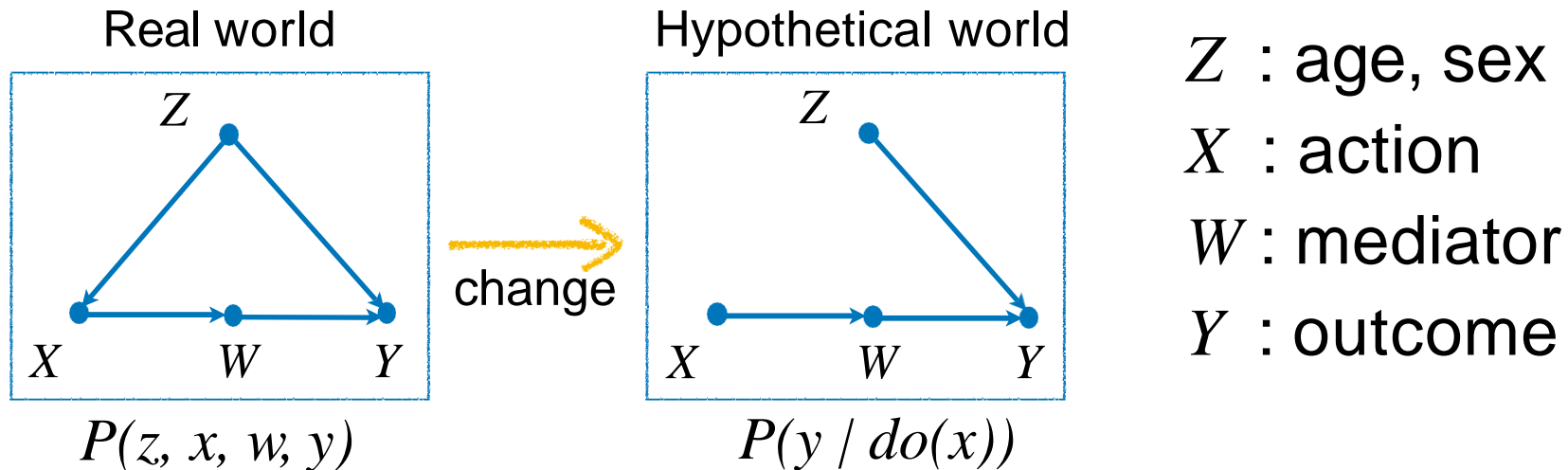
Inference

What happens when P changes?

e.g., Infer whether less people would **get cancer**
if we **ban smoking**.

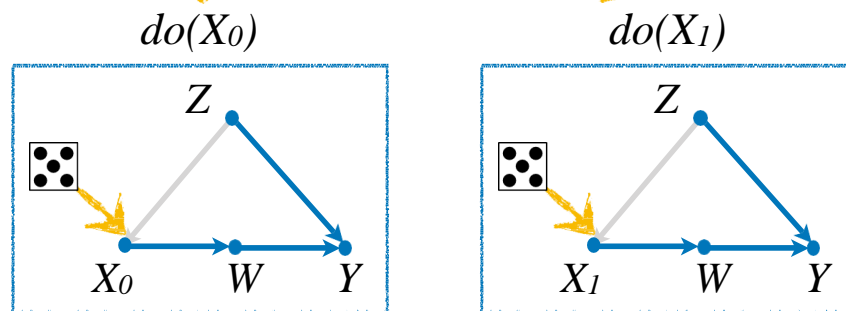
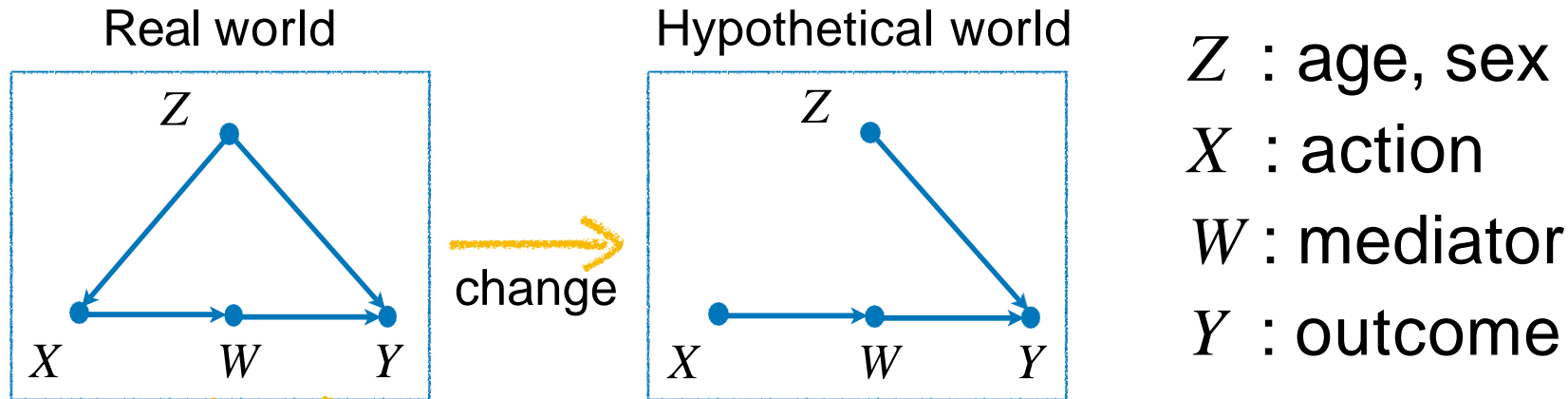
$Q = P(\text{Cancer} = \text{true} \mid \text{do}(\text{Smoking} = \text{no}))$ Not an aspect of P .

The Challenge of Causal Inference



- Goal: how much Y changes with X if we vary X between two different constants free of the influence of Z .
- These variations are called causal effects!

Method for Computing Causal Effects: Randomized Experiments



Randomization:

$$P(y \mid do(X_0))$$

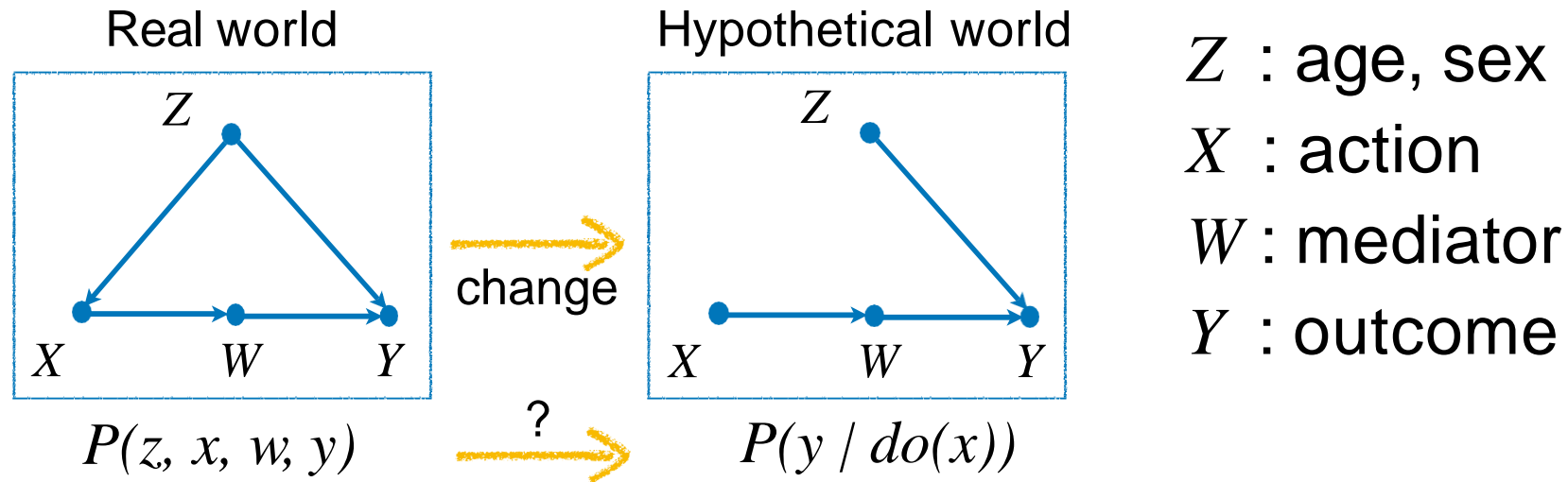
$$P(y \mid do(X_1))$$

Often we cannot do this:

How do we force people to smoke (and wait 20 years
For them to die or not)

How can we change cholesterol levels...

Computing Causal Effects (I2) from Observational Data (I1)



Questions:

- * What is the relationship between $P(z, x, w, y)$ and $P(y / do(x))$?
- * Is $P(y / do(x)) = P(y / x)$?

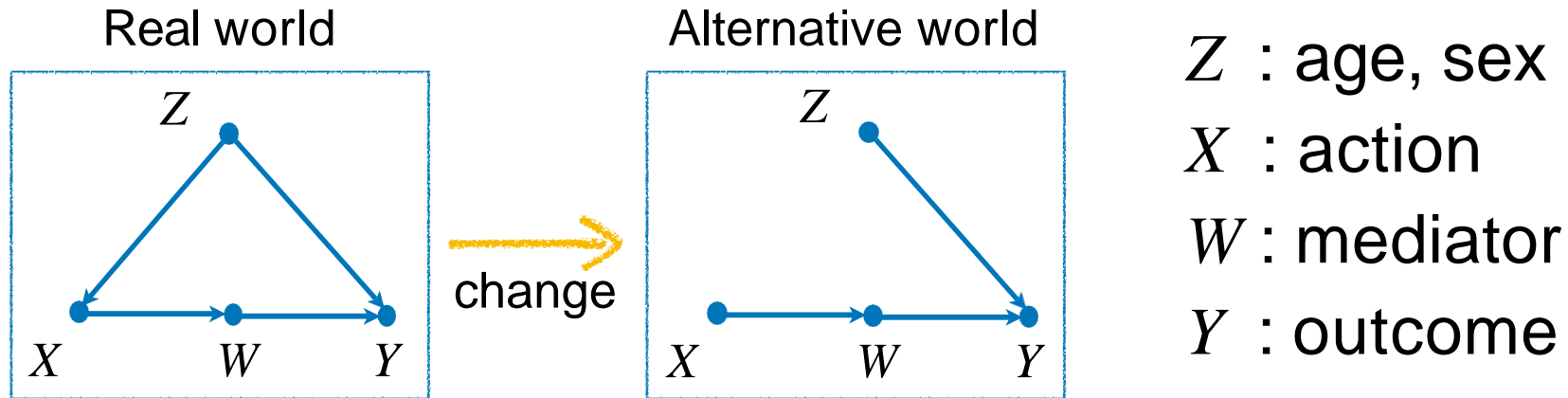
Causal Effects (formal)

Causal Effect (Def. 3.2.1 [C]):

Given two disjoint sets of variables, X and Y , the **causal effect** of X on Y , denoted as $P(\mathbf{y} / do(\mathbf{x}))$, is a function from X to the space of probability distributions of Y .

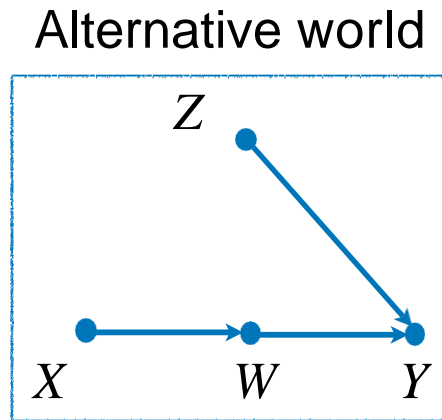
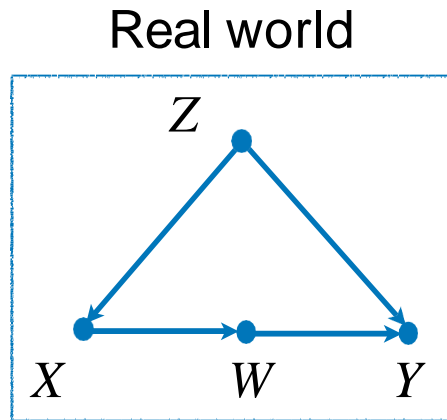
For each realization x of X , $P(\mathbf{y} / do(\mathbf{x}))$ is the probability $Y = \mathbf{y}$ induced by **deleting** from the model all equations corresponding to variables in X and **substituting** $X = \mathbf{x}$ in the remaining equations.

Computing Causal Effects from Observational Data



$$M = \begin{cases} Z = f_Z(u_z) \\ X = f_X(z, u_x) \\ W = f_W(x, u_w) \\ Y = f_Y(w, z, u_y) \end{cases} \xrightarrow{\text{do}(X=x)} M_x = \begin{cases} Z = f_Z(u_z) \\ X = f_X(z, u_x) & X = x \\ W = f_W(x, u_w) \\ Y = f_Y(w, z, u_y) \end{cases}$$

Computing Causal Effects from Observational Data



Z : age, sex

X : action

W : mediator

Y : outcome

change

$$P(v) =$$

$$P(z) \times P(x / z) \times P(w / x) \times P(y / w, z)$$

$$P_x(v) =$$

$do(X=x)$

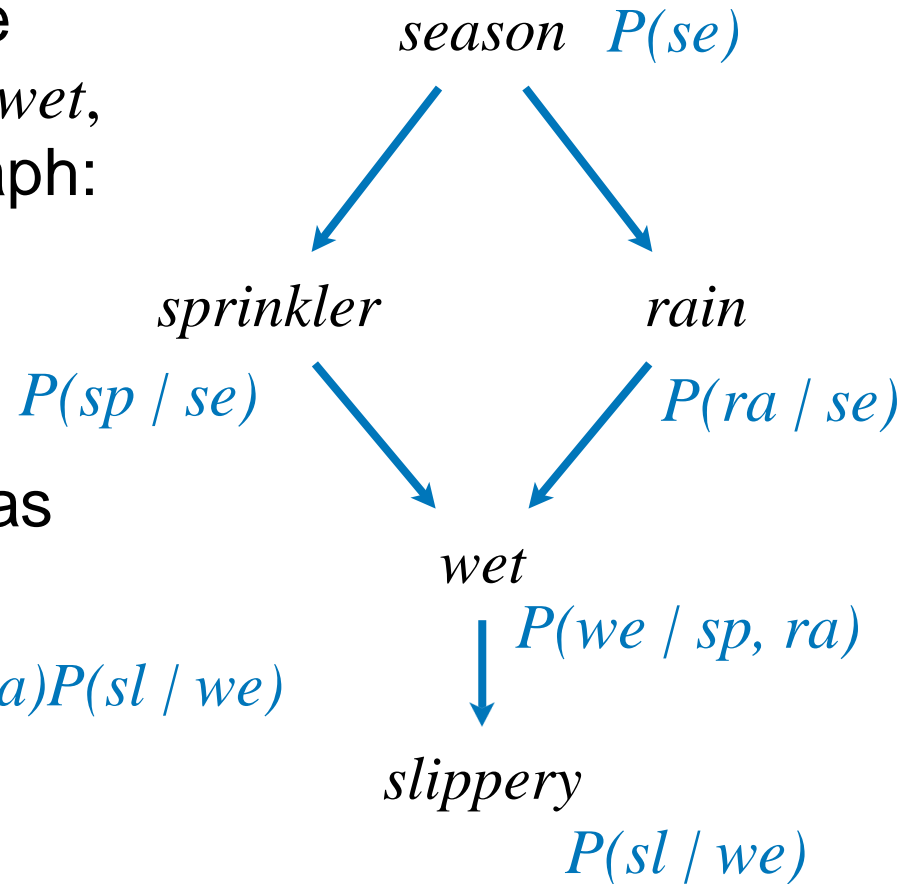
$$P(z) \times \cancel{P(x/z)} \times \text{equal to 1 in } M_x \times P(w / x) \times P(y / w, z)$$

Outline (chapter 3)

- The semantic of Intervention in SCM, the do operators
- How to determine $P(Y|\text{do}(x))$ given an SCM
- The back door criterion and the adjustment formula
- Identifiability

Computing Causal Effects from Observational Data

Consider a distribution over the variables: *season*, *sprinkler*, *rain*, *wet*, and *slippery*; and the causal graph:



This distribution decomposes as

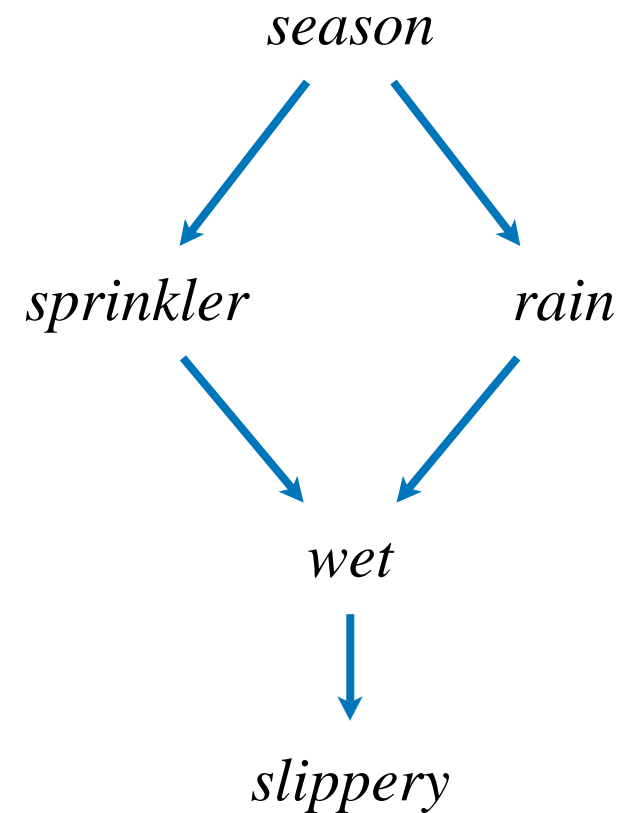
$$P(\mathbf{v}) = P(se)P(sp | se)P(ra | se)P(we | sp, ra)P(sl | we)$$

Computing Causal Effects from Observational Data

Queries:

$$Q_1 = P(\text{wet} \mid \text{Sprinkler} = \text{on})$$

$$Q_2 = P(\text{wet} \mid \text{do}(\text{Sprinkler} = \text{on}))$$

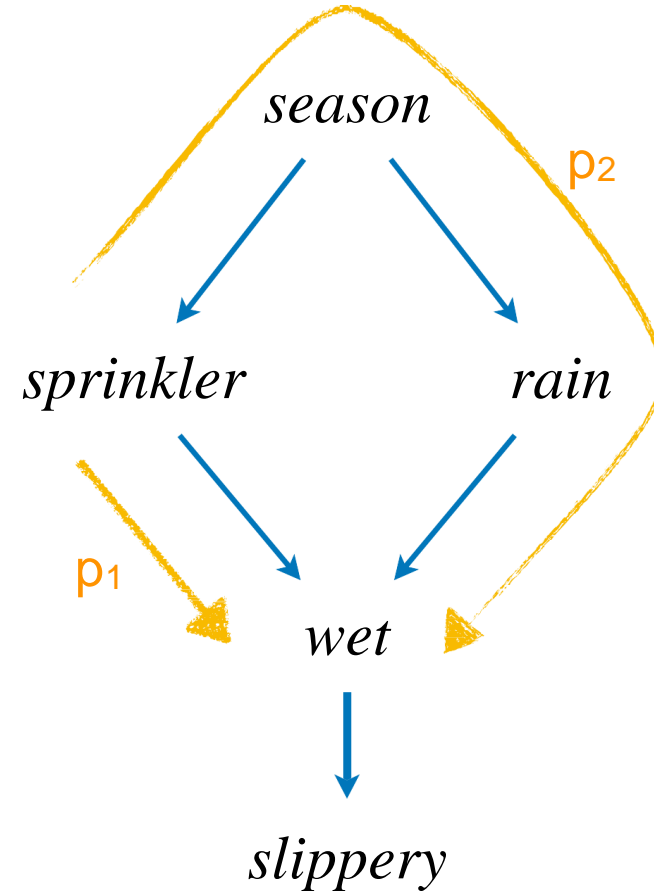


Computing Causal Effects from Observational Data

Queries:

$$Q_1 = P(\text{wet} \mid \text{Sprinkler} = \text{on}) \\ = P(p_1) + P(p_2)$$

$$Q_2 = P(\text{wet} \mid \text{do}(\text{Sprinkler} = \text{on}))$$



Computing Causal Effects from Observational Data

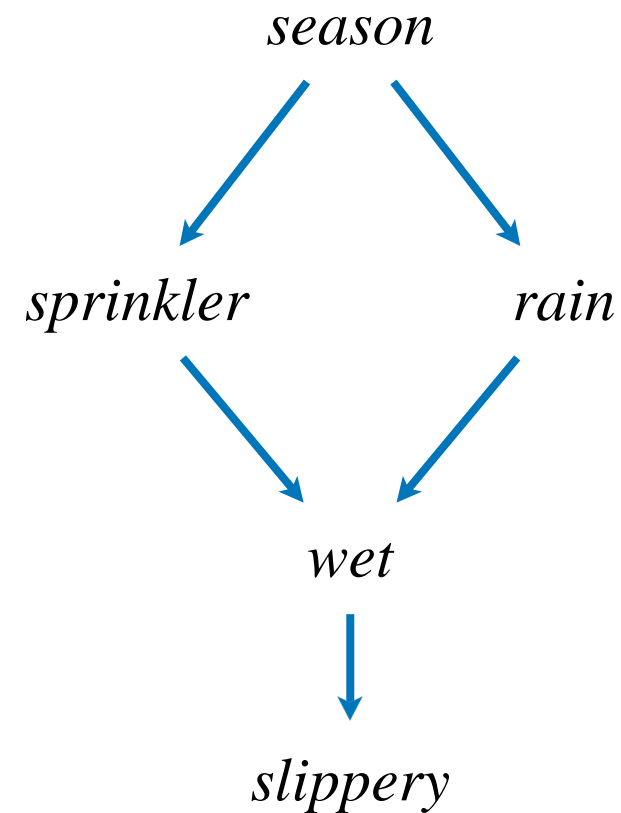
Queries:

$$Q_1 = P(\text{wet} \mid \text{Sprinkler} = \text{on})$$

$$= \frac{\sum_{se,ra} P(\text{we} \mid \text{Sp} = \text{on}, \text{ra})P(\text{Sp} = \text{on} \mid \text{se})P(\text{ra} \mid \text{se})P(\text{se})}{\sum_{se} P(\text{Sp} = \text{on} \mid \text{se})P(\text{se})}$$

$$Q_2 = P(\text{wet} \mid \text{do}(\text{Sprinkler} = \text{on}))$$

You can do algorithm bucket elimination to infer Q1.



Computing Causal Effects from Observational Data

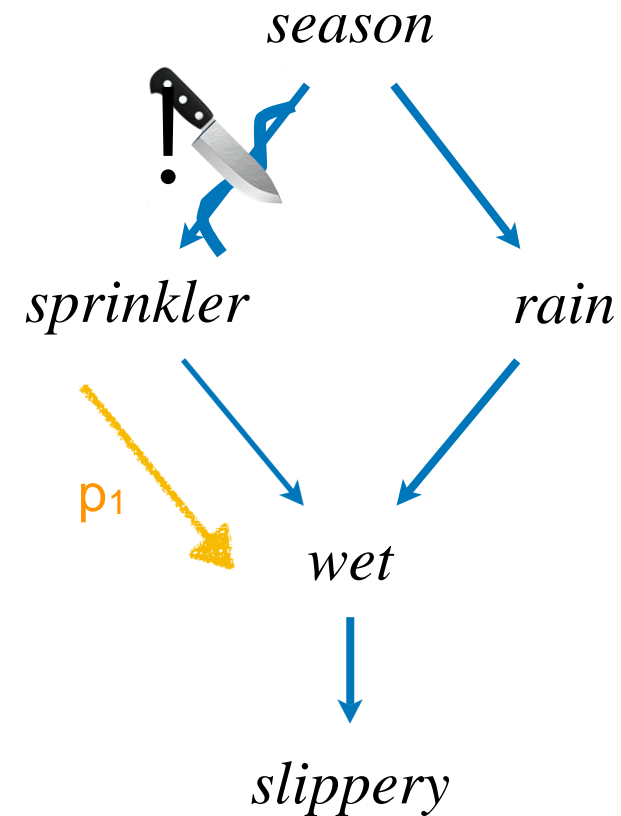
Queries:

$$Q_1 = P(\text{wet} \mid \text{Sprinkler} = \text{on})$$

$$= \frac{\sum_{se,ra} P(\text{wet} \mid \text{Sp} = \text{on}, \text{ra}) P(\text{Sp} = \text{on} \mid se) P(\text{ra} \mid se) P(se)}{\sum_{se} P(\text{Sp} = \text{on} \mid se) P(se)}$$

$$Q_2 = P(\text{wet} \mid \text{do}(\text{Sprinkler} = \text{on}))$$

You can do algorithm bucket elimination to infer Q2.



Computing Causal Effects from Observational Data

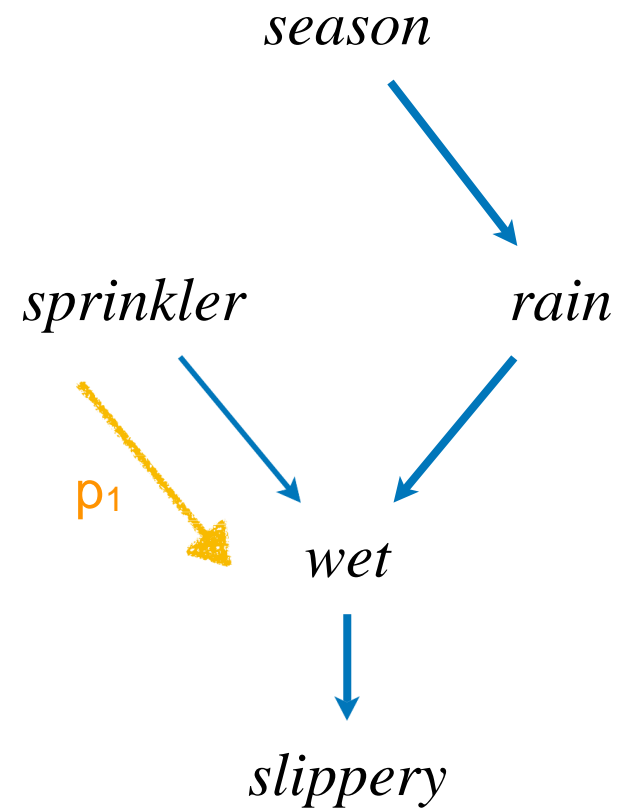
Queries:

$$Q_1 = P(\text{wet} \mid \text{Sprinkler} = \text{on})$$

$$= \frac{\sum_{se,ra} P(\text{we} \mid \text{Sp} = \text{on}, \text{ra}) P(\text{Sp} = \text{on} \mid \text{se}) P(\text{ra} \mid \text{se}) P(\text{se})}{\sum_{se} P(\text{Sp} = \text{on} \mid \text{se}) P(\text{se})}$$

$$Q_2 = P(\text{wet} \mid \text{do}(\text{Sprinkler} = \text{on})) \\ = P(p_1)$$

You can do algorithm bucket elimination to infer Q1.



Computing Causal Effects from Observational Data

Queries:

$$Q_1 = P(\text{wet} \mid \text{Sprinkler} = \text{on})$$

$$= \frac{\sum_{se,ra} P(\text{we} \mid \text{Sp} = \text{on}, ra) P(\text{Sp} = \text{on} \mid se) P(ra \mid se) P(se)}{\sum_{se} P(\text{Sp} = \text{on} \mid se) P(se)}$$

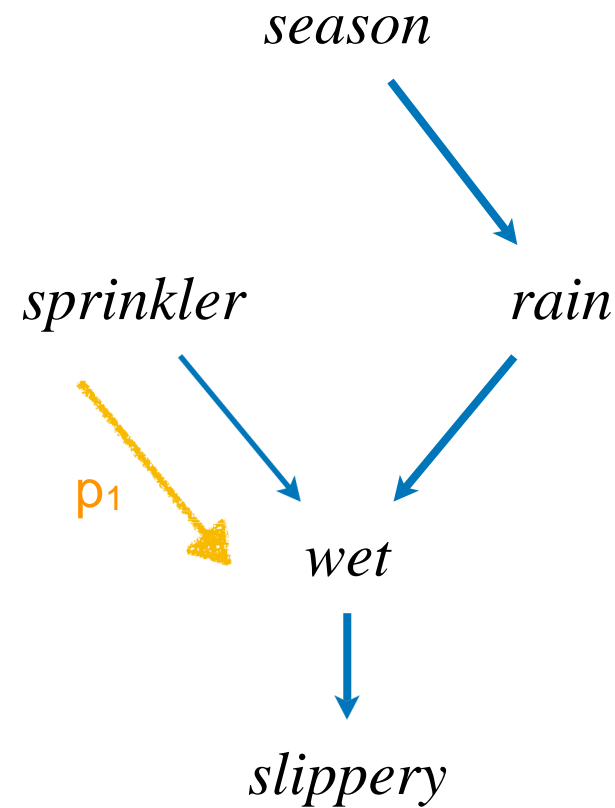
$$Q_2 = P(\text{wet} \mid \text{do}(\text{Sprinkler} = \text{on}))$$

$$= \frac{\sum_{se,ra} P(\text{we} \mid \text{Sp} = \text{on}, ra) P(\text{Sp} = \text{on}) P(ra \mid se) P(se)}{P(\text{Sp} = \text{on})}$$

$$= \sum_{se,ra} P(\text{we} \mid \text{Sp} = \text{on}, ra) P(ra \mid se) P(se)$$

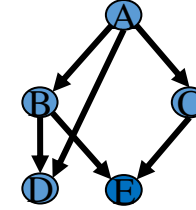
equal to 1

You can do algorithm bucket elimination to infer Q2.



Bucket elimination

Algorithm *BE-bel* (Dechter 1996)



$$P(A | E = 0) = \alpha \sum_{E=0, D, C, B} P(A) \cdot P(B | A) \cdot P(C | A) \cdot P(D | A, B) \cdot P(E | B, C)$$

$\sum_b \prod$ ← Elimination operator

bucket B:

$$P(b|a) \quad P(d|b,a) \quad P(e|b,c)$$

bucket C:

$$P(c|a) \quad \lambda^B(a, d, c, e)$$

bucket D:

$$\lambda^C(a, d, e)$$

bucket E:

$$e=0 \quad \lambda^D(a, e)$$

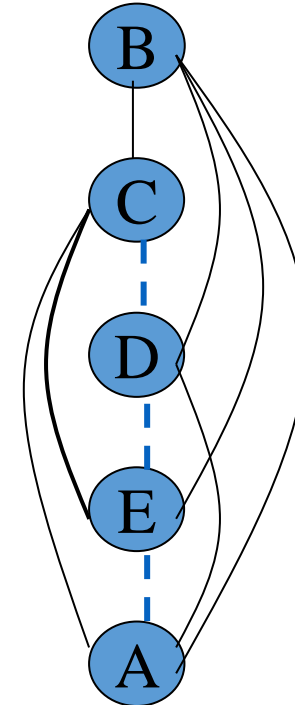
bucket A:

$$P(a) \quad \lambda^E(a)$$

$$P(a, e=0)$$

$$P(e=0)$$

$W^*=4$
"induced width"
(max clique size)



$$P(a|e=0) = \frac{P(a, e=0)}{P(e=0)}$$

Truncated Factorization Product (Operationalizing Interventions)

Corollary (Truncated Factorization, Manipulation Thm., G-comp.):

The distribution generated by an intervention $do(\mathbf{X}=\mathbf{x})$ (in a [Markovian](#) model M) is given by the truncated factorization:

$$P(\mathbf{v} \mid do(\mathbf{X})) = \prod_{\{V_i \in \mathbf{V} \setminus \mathbf{X}\}} P(v_i \mid pa_i) \Big|_{\mathbf{X}=\mathbf{x}}$$

Truncated Factorization Formula

The truncated product,

$$P(\mathbf{v} \mid do(\mathbf{x})) = \prod_{\{V_i \in \mathbf{V} \setminus \mathbf{X}\}} P(v_i \mid pa_i) \Big|_{\mathbf{X}=\mathbf{x}}$$

can be rewritten as:

$$P(\mathbf{v} \mid do(\mathbf{x})) = \frac{P(\mathbf{v})}{P(\mathbf{x} \mid pa_{\mathbf{x}})} \Big|_{\mathbf{X}=\mathbf{x}}$$

Also equivalent to:

$$P(\mathbf{v} \mid do(\mathbf{x})) = P(\mathbf{v} \mid \mathbf{x}, pa_{\mathbf{x}}) P(pa_{\mathbf{x}}) \Big|_{\mathbf{X}=\mathbf{x}}$$

The transformation between the observation and interventional distributions can be seen as a re-weighting process.

Intervention vs. Conditioning, The Ice-Cream Story

Conditioning $P(Y=y|X=x)$

Intervening $P(Y=y| \text{do}(X=x))$

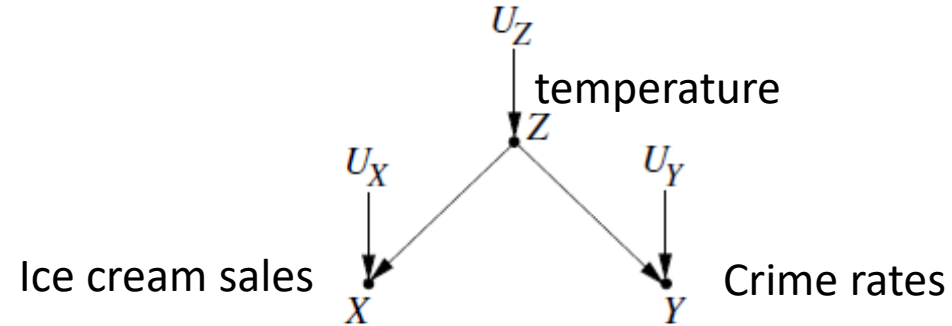
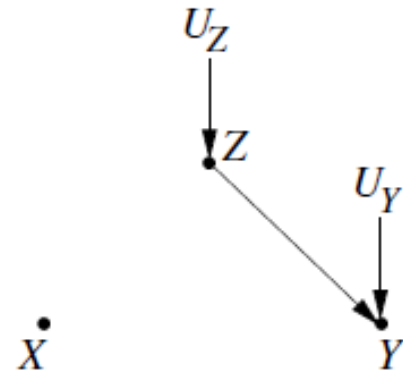


Figure 3.1: A graphical model representing the relationship between temperature (Z), ice cream sales (X), and crime rates (Y)

When we intervene to fix a value of a variable, We curtail the natural tendencies of the variable to vary In response to other variables in nature.



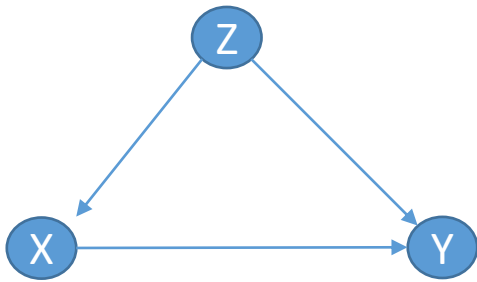
- **This corresponds to a surgery of the model**
- i.e. varying Z will not affect X
- **Intervention depends on the structure of the graph.**

Figure 3.2: A graphical model representing an intervention on the model in Figure 3.1 that lowers ice cream sales

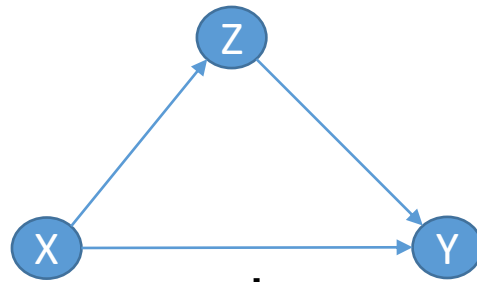
Intervention vs Conditioning, The Surgery Operation

Conditioning $P(Y=y|X=x)$

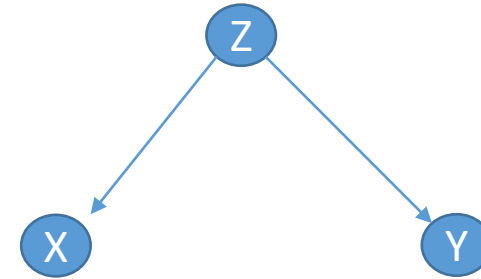
The Simpson story



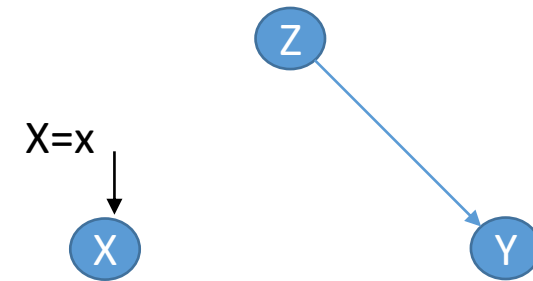
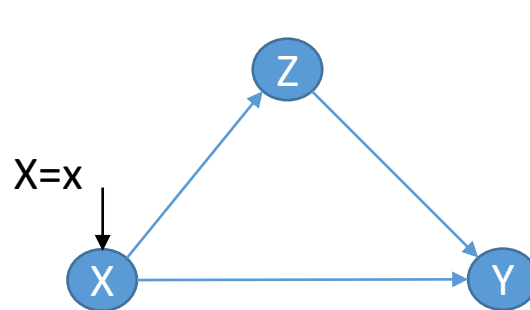
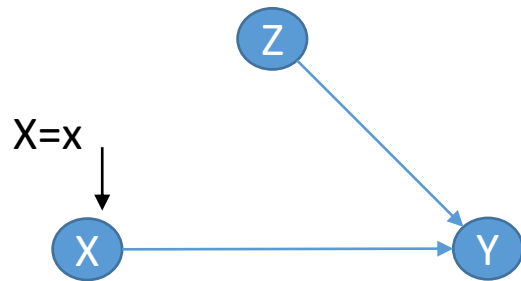
The blood pressure story



The ice-cream story



Intervening $P(Y=y| \text{do}(X=x))$



Intervention vs. Conditioning...

In notation, we distinguish between cases where a variable X takes a value x naturally and cases where we fix $X = x$ by denoting the latter $do(X = x)$. So $P(Y = y|X = x)$ is the probability that $Y = y$ conditional on finding $X = x$, while $P(Y = y|do(X = x))$ is the probability that $Y = y$ when we intervene to make $X = x$. In the distributional terminology, $P(Y = y|X = x)$ reflects the population distribution of Y among individuals whose X value is x . On the other hand, $P(Y = y|do(X = x))$ represents the population distribution of Y if *everyone in the population* had their X value fixed at x . We similarly write $P(Y = y|do(X = x), Z = z)$ to denote the conditional probability of $Y = y$, given $Z = z$, in the distribution created by the intervention $do(X = x)$.

Do operation and graph surgery can help determine causal effect

We make an assumption that intervention has no side-effect. Namely, assigning a variable by intervention does not affect other variables in a direct way. Truncation is local

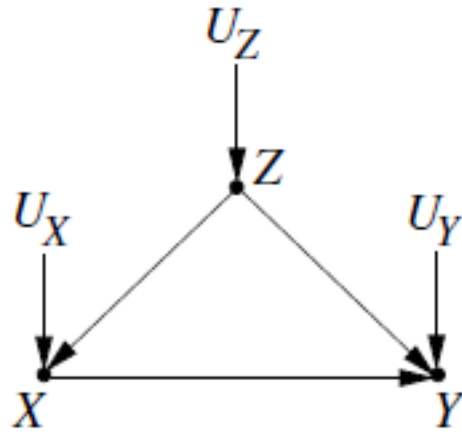
The Adjustment Formula

To find out how effective the drug is in the population, we imagine a hypothetical intervention by which we administer the drug uniformly to the entire population and compare the recovery rate to what would obtain under the complementary intervention, where we prevent everyone from using the drug.

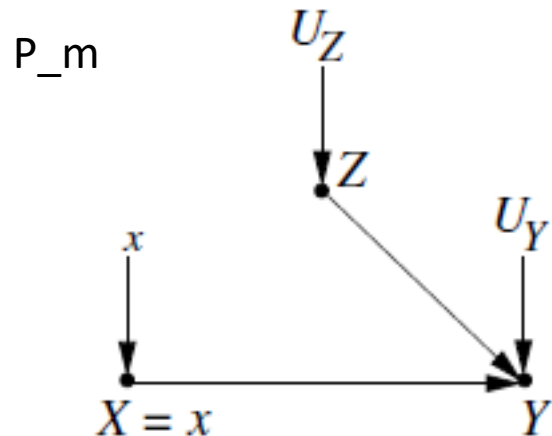
We want to estimate the “causal effect difference,” or “average causal effect” (ACE).

$$P(Y = 1 | do(X = 1)) - P(Y = 1 | do(X = 0)) \quad (3.1)$$

Definition of Intervention and Graph Surgery: The Adjustment Formula



- We simulate the intervention in the form of a graph surgery.
- The causal effect $P(Y = y | do(X = x))$ equals to the conditional probability $P_m(Y = y | X = x)$ that prevails in the manipulated model of the figure below



Important: the random functions for Z and Y remain invariant

$$P_m(Y = y | Z = z, X = x) = P(Y = y | Z = z, X = x) \quad \text{and} \quad P_m(Z = z) = P(Z = z)$$

The Adjustment Formula



$$P(Y = y|do(X = x)) = P_m(Y = y|X = x) \quad (\text{by definition}) \quad (3.2)$$

$$= \sum_z P_m(Y = y|X = x, Z = z)P_m(Z = z|X = x) \quad (3.3)$$

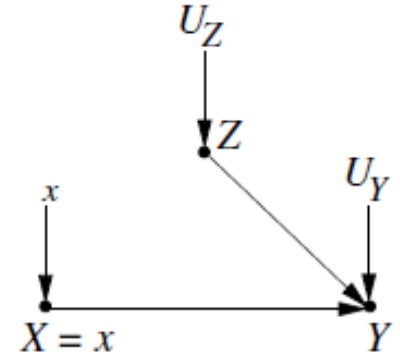
$$= \sum_z P_m(Y = y|X = x, Z = z)P_m(Z = z) \quad (3.4)$$

Equation (3.3) is obtained from Bayes' rule by conditioning on and summing over all values of $Z = z$ (as in Eq. (1.19)), while (Eq. 3.4) makes use of the independence of Z and X in the modified model.

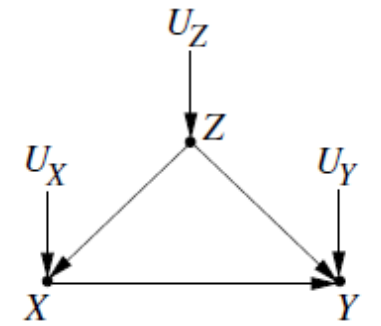
Finally, using the invariance relations, we obtain a formula for the causal effect, in terms of preintervention probabilities:

$$P(Y = y|do(X = x)) = \sum_z P(Y = y|X = x, Z = z)P(Z = z) \quad (3.5)$$

Equation (3.5) is called the *adjustment formula* and as you can see, it computes the association between X and Y for each value z of Z , then averages over those values. This procedure is referred to as “adjusting for Z ” or “controlling for Z .”



The Adjustment Formula (in the Simpson story)



$$P(Y = y|do(X = x)) = \sum_z P(Y = y|X = x, Z = z)P(Z = z) \quad (3.5)$$

The right hand-side can be estimated from the data since it has only conditional probabilities.

If we had a randomized controlled experiments on X (taking the drug) we would not need adjustment
Because the data is already generated from the manipulated distribution. Namely it will yield $P(Y=y|do(x))$
From the data of the randomized experiment.

In practice adjustment is sometime used in randomized experiments to reduce sampling variations (Cox 1958).

(This means: If the input is sampled from the **intervened upon** joint distribution over X,Y and Z we can estimate the $P(y|x)$ directly. Or, we can first estimate $P(y|x,z)$ and also $P(z)$ and perform the summation.)

In the Simpson example:

Table 1.1 Results of a study into a new drug, with gender taken into account

| | Drug | No drug |
|---------------|--------------------------------|--------------------------------|
| Men | 81 out of 87 recovered (93%) | 234 out of 270 recovered (87%) |
| Women | 192 out of 263 recovered (73%) | 55 out of 80 recovered (69%) |
| Combined data | 273 out of 350 recovered (78%) | 289 out of 350 recovered (83%) |

$$P(Y = 1|do(X = 1)) = P(Y = 1|X = 1, Z = 1)P(Z = 1) + P(Y = 1|X = 1, Z = 0)P(Z = 0)$$

Substituting the figures given in Table 1.1 we obtain

$$P(Y = 1|do(X = 1)) = \frac{0.93(87 + 270)}{700} + \frac{0.73(263 + 80)}{700} = 0.832$$

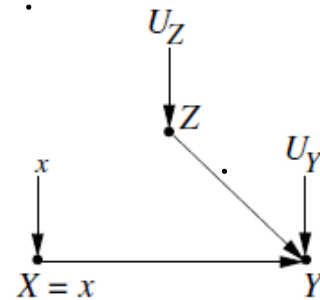
while, similarly,

$$P(Y = 1|do(X = 0)) = \frac{0.87(87 + 270)}{700} + \frac{0.69(263 + 80)}{700} = 0.7818$$

We get that the Average Causal Effect (ACE):

$$ACE = P(Y = 1|do(X = 1)) - P(Y = 1|do(X = 0)) = 0.832 - 0.7818 = 0.0502$$

A more informal interpretation of ACE is that it is the difference in the fraction of the population that would recover if everyone took the drug compared to when no one takes the drug.



The Blood Pressure Example

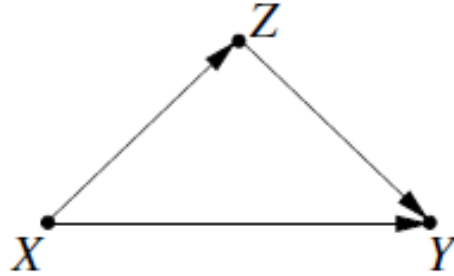


Figure 3.5: A graphical model representing the effects of a new drug, with X representing drug usage, Y representing recovery, and Z representing blood pressure (measured at the end of the study). Exogenous variables are not shown in the graph, implying that they are mutually independent

$$P(Y=y \mid \text{do}(X=x)) = ?$$

Here the “surgery on X changes nothing. So,

This means that no surgery is required; the conditions under which data were obtained were such that treatment was assigned “as if randomized.” If there was a factor that would make subjects prefer or reject treatment, such a factor should show up in the model; the absence of such a factor gives us the license to treat X as a randomized treatment.

$$P(Y = y \mid \text{do}(X = x)) = P(Y = y \mid X = x),$$

To Adjust or not to Adjust?

Rule 1 (The Causal Effect Rule) *Given a graph G in which a set of variables PA are designated as the parents of X , the causal effect of X on Y is given by*

$$P(Y = y|do(X = x)) = \sum_z P(Y = y|X = x, PA = z)P(PA = z) \quad (3.6)$$

Where z ranges over all the combinations of values that the variables that PA take.

So, the causal graph helps determine the parents PA !

But, in many cases some of the parents are unobserved so we cannot perform the calculation.

Luckily we can often adjust for other variables substituting for the unmeasured variables in $PA(X)$, and this can be decided via the graph.

Multiple Interventions, the Truncated Product Rule

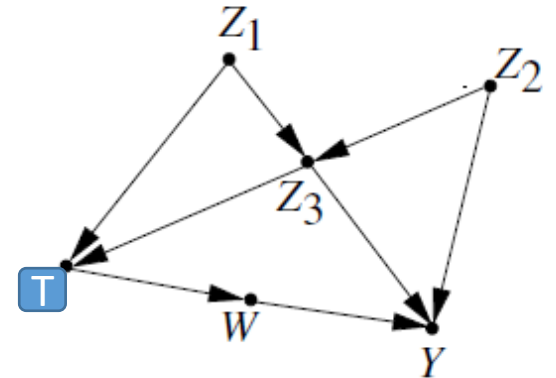
Often we have multiple interventions that may not correspond to disconnected variables. We will use the product decomposition. We write **the product truncated formula**

$$P(x_1, x_2, \dots, x_n | do(x)) = \prod_i P(x_i | pa_i) \quad \text{for all } i \text{ with } X_i \text{ not in } X.$$

Example:

$$P(z_1, z_2, w, y | do(T = t, Z_3 = z_3)) = P(z_1)P(z_2)P(w|t)P(y|w, z_3, z_2)$$

where we have deleted the factors $P(t|z_1, z_3)$ and $P(z_3|z_1, z_2)$ from the product.



Multiple Interventions and the Truncated Product Rule

preintervention distribution in the model of Figure 3.3 is given by

$$P(x, y, z) = P(z)P(x|z)P(y|x, z)$$

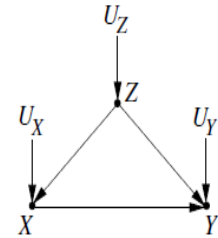


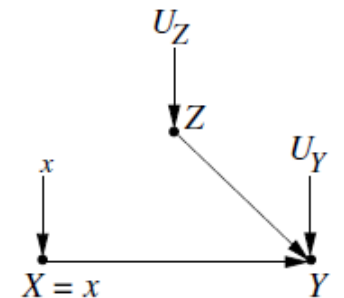
Figure 3.3: A graphical model representing the effects of a new drug, with Z representing gender, X standing for drug usage, and Y standing for recovery

whereas the postintervention distribution, governed by the model of Figure 3.4 is given by the product

$$P(z, y|do(x)) = P_m(z)P_m(y|x, z) = P(z)P(y|x, z) \quad (3.9)$$

with the factor $P(x|z)$ purged from the product, since X becomes parentless as it is fixed at $X = x$. This coincides with the adjustment formula, because to evaluate $P(y|do(x))$ we need to marginalize (or sum) over z , which gives

$$P(y|do(x)) = \sum_z P(z)P(y|x, z)$$



Outline (chapter 3)

- The semantic of Intervention in SCM, the do operators
- How to determine $P(Y | do(x))$ given an SCM
- **The identification problem**
- The back door criterion and the adjustment formula

How Can We Do It? partial model + data

Input:

- Causal graph (no parameters)
- Data (observational)

Output:

- Causal effect

Key observations:

- This may not always be possible
- When it is possible, we say that the causal effect is **identifiable**
- Identifiability depends on **type of causal graph** and **available data**
- Several criteria for deciding identifiability: some are **complete**, some are not

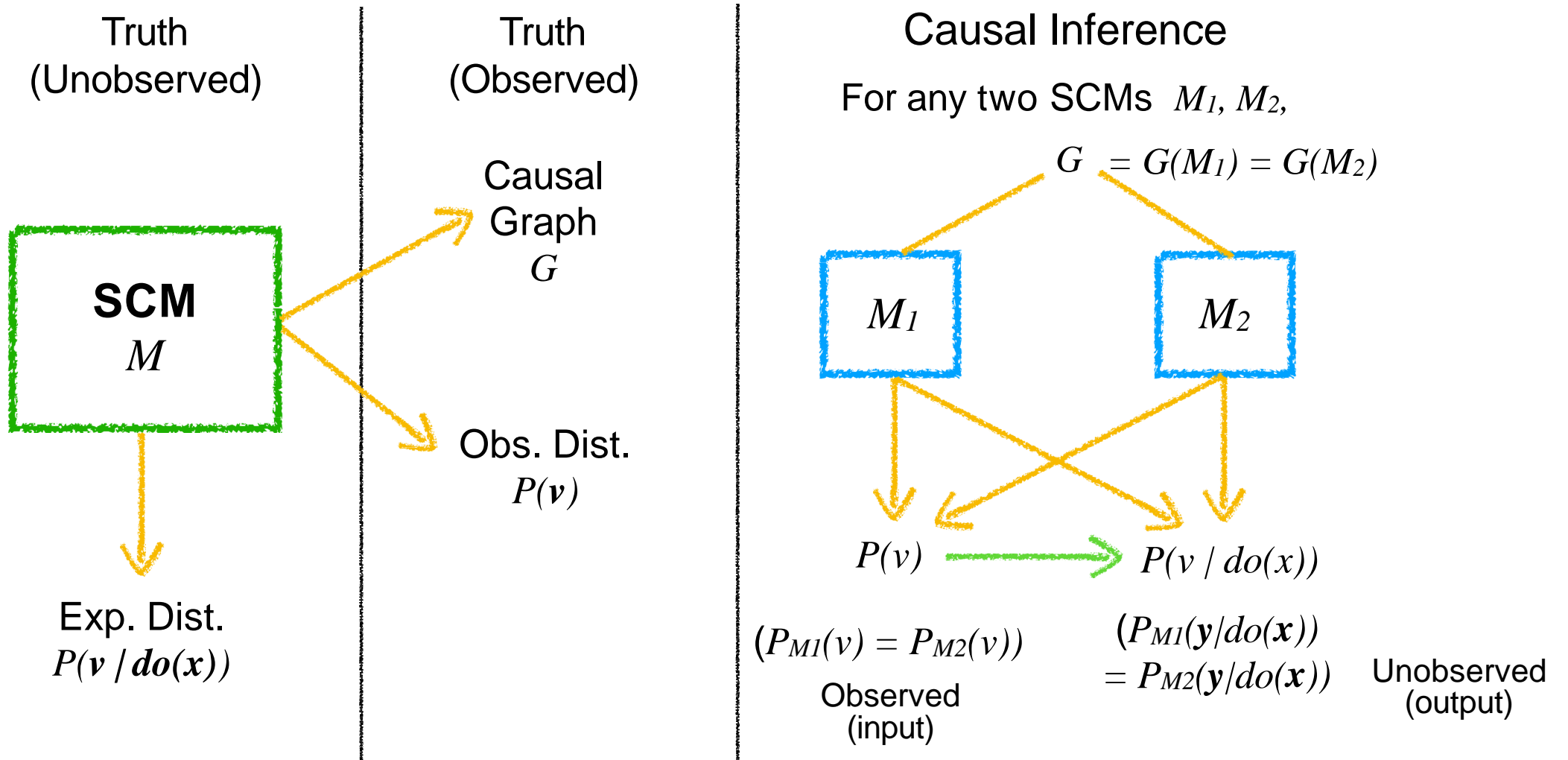
The Identification Problem

Causal Effect Identifiability (Def. 3.2.2)

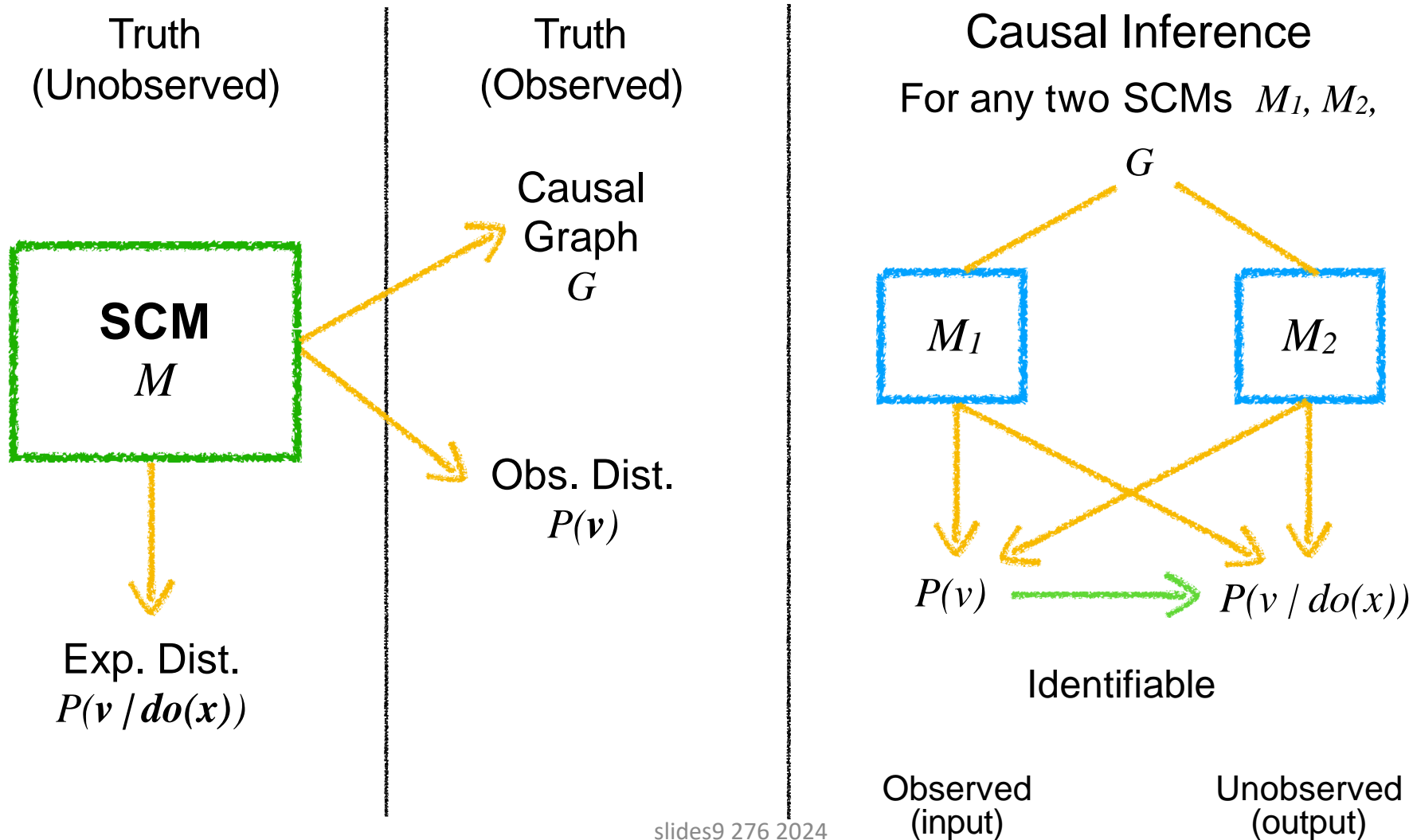
The **causal effect** of X on Y is said to be **identifiable** from a causal diagram G if the quantity $P(\mathbf{y} / do(\mathbf{x}))$ can be computed uniquely from a **positive** probability of the observed variables.

That is, if for every pair of models M_1 and M_2 inducing G , $P_{M_1}(\mathbf{y} / do(\mathbf{x})) = P_{M_2}(\mathbf{y} / do(\mathbf{x}))$, whenever $P_{M_1}(\mathbf{v}) = P_{M_2}(\mathbf{v}) > 0$.

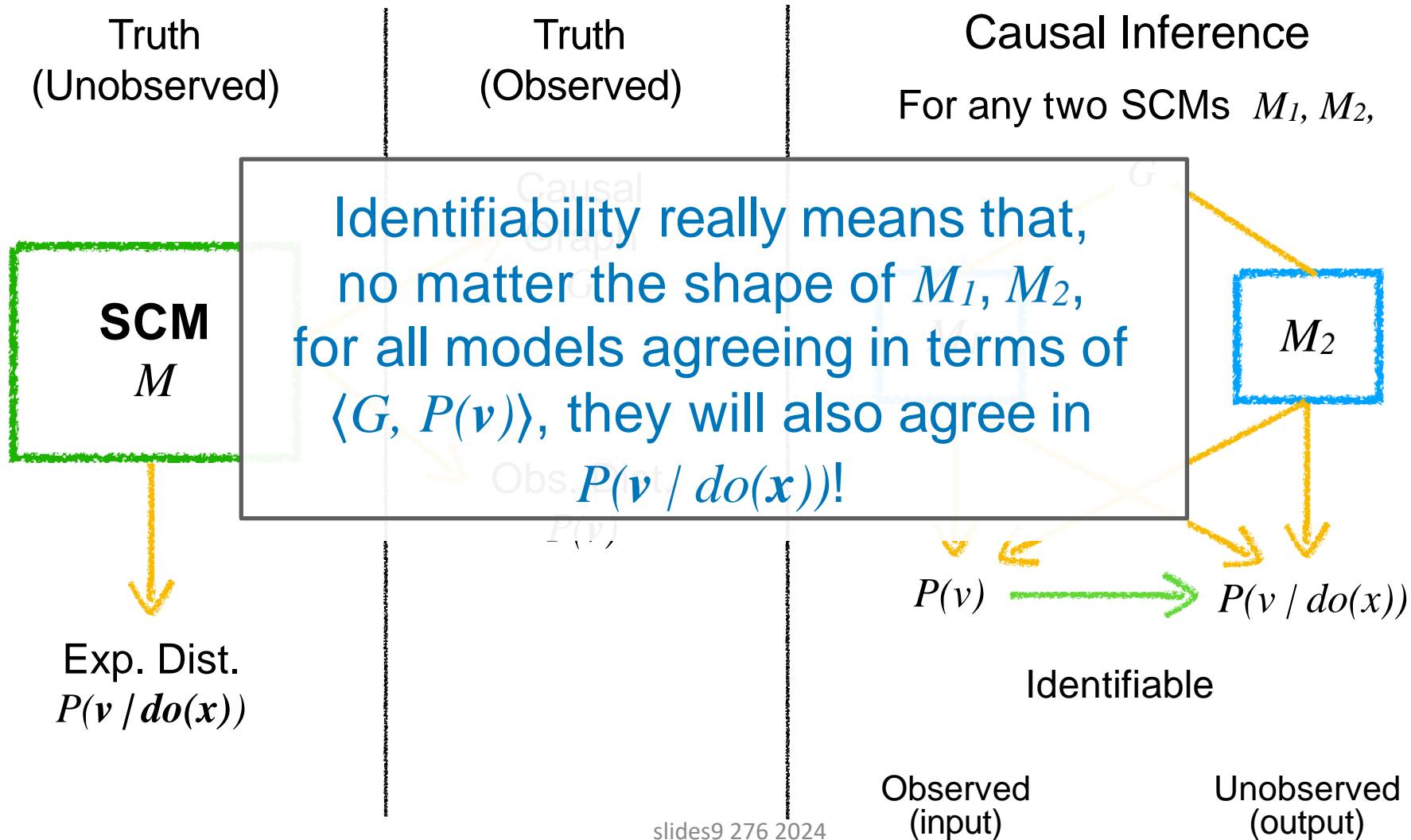
The Identification Problem (II)



The Identification Problem (II)



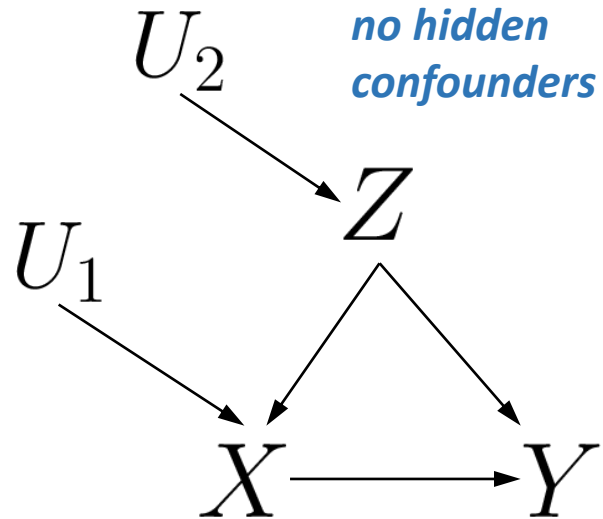
The Identification Problem (II)



Types of Causal Graphs hidden variables are roots

Markovian Model

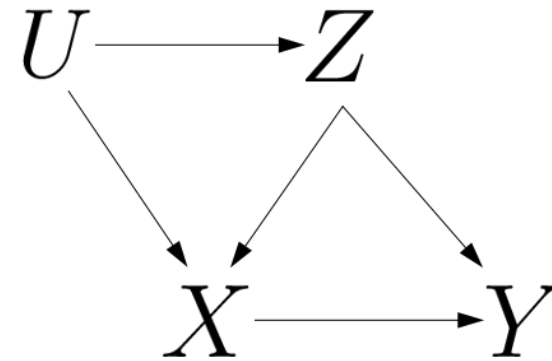
each hidden variable has **at most** one child



causal effect **always** identifiable

Semi-Markovian Model

some hidden variable has **more than** one child

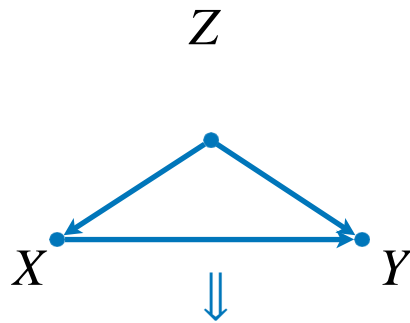


$Pr(y_x)$ identifiable $Pr(x_z)$ not identifiable

causal effect **not always** identifiable

Example. Identifiable Effect

- Consider any two pair of models compatible with the following graph and the same observational distribution $P(\mathbf{v})$:

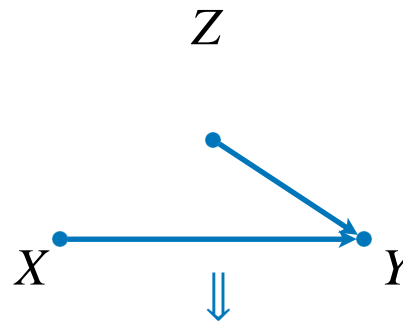


$$P(\mathbf{v}) = P(z)P(x/z)P(y/x,z)$$

$$M^{(1)} = \begin{cases} Z \leftarrow f_z^{(1)}(u_z) \\ X \leftarrow f_x^{(1)}(z, u_x) \\ Y \leftarrow f_y^{(1)}(x, z, u_y) \end{cases}$$

$do(x)$
↳

$$M^{(2)} = \begin{cases} Z \leftarrow f_z^{(2)}(u_z) \\ X \leftarrow f_x^{(2)}(z, u_x) \\ Y \leftarrow f_y^{(2)}(x, z, u_y) \end{cases}$$



$$P(\mathbf{v}/do(x)) = P(z)P(y/x,z) \Rightarrow P(y | do(x)) = \sum P(z)P(y | x, z)$$

$$M^{(1)} = \begin{cases} Z \leftarrow f_z^{(1)}(u_z) \\ X \leftarrow x \\ Y \leftarrow f_y^{(1)}(x, z, u_y) \end{cases}$$

$$M^{(2)} = \begin{cases} Z \leftarrow f_z^{(2)}(u_z) \\ X \leftarrow x \\ Y \leftarrow f_y^{(2)}(x, z, u_y) \end{cases}$$

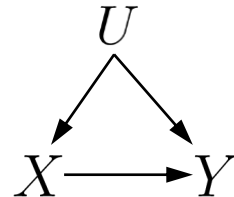
No matter what the specific functions or $P(u)$ are, as long as M_1, M_2 agree in $\langle G, P(\mathbf{v}) \rangle$, they will also agree in $P(z)$ and $P(y/x,z)$, hence in $P(\mathbf{v} / do(x))!$

$$PR(y_X) = P(y | do(x))$$

Why Causal Effect May Not Be Identifiable?

Consider the model

If X and Y are off, cannot tell if X turned off Y or if U turned off both



- $Pr(y|x, u) = Pr(\bar{y}|\bar{x}, u) = 1$ and $Pr(x|u) = q$
- $Pr(\bar{x}|\bar{u}) = Pr(\bar{y}|X, \bar{u}) = 1$
- $Pr(u) = p$

Causal effect (using truncated formula of interventional distribution)

$$\begin{aligned}
 Pr(y|do(x)) &= Pr_x(y) \\
 &= Pr_x(u, y) + Pr_x(\bar{u}, y) \\
 &= Pr(u)Pr(y|x, u) + Pr(\bar{u})Pr(y|x, \bar{u}) \\
 &= p(1) + (1 - p)(0) = p
 \end{aligned}$$

Data generated by the model

| U | X | Y | frequency |
|-----------|-----------|-----------|------------|
| u | x | y | pq |
| u | \bar{x} | \bar{y} | $p(1 - q)$ |
| \bar{u} | \bar{x} | \bar{y} | $1 - p$ |

If U is hidden, we see this data

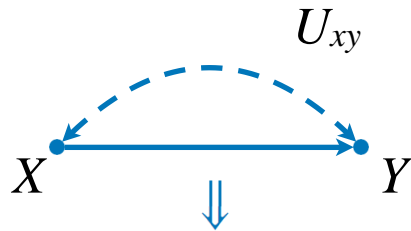
| X | Y | frequency |
|-----------|-----------|-----------|
| x | y | pq |
| \bar{x} | \bar{y} | $1 - pq$ |

Cannot recover the causal effect...

Example: if $pq=.14$ then $p=.7, q=.2$ and $p=.2, q=.7$ are solutions, but with different causal effects

Example. Non-identifiable Effect

- Consider the pair of models compatible with the following graph G and observational distribution $P(\mathbf{v})$:



$$P(\mathbf{v}) = \sum_{u_{xy}} P(y|x, u_{xy})P(x|u_{xy})P(u_{xy})$$

$$M^{(1)} = \begin{cases} X \leftarrow U_{xy} \\ Y \leftarrow (X \oplus U_{xy}) \vee U_y \end{cases}$$

$$M^{(2)} = \begin{cases} X \leftarrow U_{xy} \\ Y \leftarrow U_y \end{cases}$$

$$P^{(j)}(U_i = 1) = 1/2, \\ i = \{x, y, xy\}, j = \{1, 2\}$$

| U_y | U_{xy} | X | $Y^{(1)}$ | $Y^{(2)}$ | $P(\mathbf{v}, \mathbf{u})$ |
|-------|----------|-----|-----------|-----------|-----------------------------|
| 0 | 0 | 0 | 0 | 0 | 1/8 |
| 0 | 1 | 1 | 0 | 0 | 1/8 |
| 1 | 0 | 0 | 1 | 1 | 1/8 |
| 1 | 1 | 1 | 1 | 1 | 1/8 |
| 0 | 0 | 0 | 0 | 0 | 1/8 |
| 0 | 1 | 1 | 0 | 0 | 1/8 |
| 1 | 0 | 0 | 1 | 1 | 1/8 |
| 1 | 1 | 1 | 1 | 1 | 1/8 |



| X | $Y^{(1)}$ | $Y^{(2)}$ | $P^{(1)}(\mathbf{v})$ | $P^{(2)}(\mathbf{v})$ |
|-----|-----------|-----------|-----------------------|-----------------------|
| 0 | 0 | 0 | 1/4 | 1/4 |
| 0 | 1 | 1 | 1/4 | 1/4 |
| 1 | 0 | 0 | 1/4 | 1/4 |
| 1 | 1 | 1 | 1/4 | 1/4 |



They match in $P(\mathbf{v})$, that is, $P^{(1)}(\mathbf{v})=P^{(2)}(\mathbf{v})!$

Example. Non-identifiable Effect

- Consider the pair of models compatible with the following graph G and observational distribution $P(\mathbf{v})$:

U_{xy}

$X \rightarrow Y$

U_{xy}

$X \rightarrow Y$

$$P(\mathbf{v}) = \sum_{u_{xy}} P(y|x, u_{xy})P(x|u_{xy})P(u_{xy})$$

$$P(\mathbf{v} | do(x)) = \sum_{u_{xy}} P(y|x, u_{xy})P(u_{xy})$$

$$M^{(1)} = \begin{cases} X \leftarrow U_{xy} \\ Y \leftarrow (X \oplus U_{xy}) \vee U_y \end{cases}$$

$$M^{(2)} = \begin{cases} X \leftarrow U_{xy} \\ Y \leftarrow U_y \end{cases}$$

$do(x)$

\Downarrow

| U_y | U_{xy} | X | $Y^{(1)}$ | $Y^{(2)}$ | $P(\mathbf{v}, \mathbf{u} do(x))$ |
|-------|----------|-----|--------------|-----------|-------------------------------------|
| 0 | 0 | x | $0 \oplus x$ | 0 | 1/8 |
| 0 | 1 | x | $1 \oplus x$ | 0 | 1/8 |
| 1 | 0 | x | 1 | 1 | 1/8 |
| 1 | 1 | x | 1 | 1 | 1/8 |
| 0 | 0 | x | $0 \oplus x$ | 0 | 1/8 |
| 0 | 1 | x | $1 \oplus x$ | 0 | 1/8 |
| 1 | 0 | x | 1 | 1 | 1/8 |
| 1 | 1 | x | 1 | 1 | 1/8 |

\Downarrow

| $Y^{(1)}$ | $Y^{(2)}$ | $P^{(1)}(y do(x))$ | $P^{(2)}(y do(x))$ |
|-----------|-----------|--------------------|--------------------|
| 0 | 0 | 1/4 | 1/2 |
| 1 | 1 | 3/4 | 1/2 |

$$P^{(j)}(U_i = 1) = 1/2, \\ i = \{x, y, xy\}, j = \{1, 2\}$$

Even though both models induce G and have the same $P(\mathbf{v})$, the effect $P^{(1)}(y|do(x)) \neq P^{(2)}(y|do(x))!$

Let's study how to *decide*
whether a causal effect is
identifiable...

Identification in Markovian Models

Theorem. Given the causal diagram G of any Markovian model that all variables are measured, the causal effect $Q = P(\mathbf{y} / do(\mathbf{x}))$ is identifiable for every subsets of variables \mathbf{X} and \mathbf{Y} and is obtained from the truncated factorization, i.e.,

$$P(\mathbf{v} | do(\mathbf{x})) = \prod_{\{V_i \in \mathbf{V} \setminus \mathbf{X}\}} P(v_i | pa_i) \quad \text{Sum over all variables not in } \mathbf{X} \cup \mathbf{Y}$$

$$P(\mathbf{y} | do(\mathbf{x})) = \sum_{\mathbf{v} \setminus (\mathbf{x} \cup \mathbf{y})} \prod_{\{V_i \in \mathbf{V} \setminus \mathbf{X}\}} P(v_i | pa_i)$$

Adjustment by Direct Parents

Thm. Given a causal diagram G of any Markovian system, the causal quantity $Q = P(\mathbf{y} \mid do(\mathbf{x}))$ is identifiable whenever $\{X, Y, Pa_x\} \subseteq V$, that is, whenever X , Y , and all the parents of variables X are measured. The expression of Q is then obtained by adjustment for PA_x , or

$$P(\mathbf{y} \mid do(\mathbf{x})) = \sum_{pa_x} P(\mathbf{y} \mid \mathbf{x}, pa_x) P(pa_x)$$

- Quiz: 1) derive from previous slide
2) derive for non-Markovian models