Reasoning with graphical models

Slides Set 10 (part a):

Sampling Techniques for Probabilistic and Deterministic Graphical models

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(Reading" Darwiche chapter 15, related papers)

Overview

1. Basics of sampling
2. Importance Sampling
3. Markov Chain Monte Carlo: Gibbs Sampling
4. Sampling in presence of Determinism
5. Rao-Blackwellisation, cutset sampling

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Types of queries

Max-Inference	$f(\mathbf{x}^*) = \max_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Sum-Inference	$Z = \sum_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Mixed-Inference	$f(\mathbf{x}_M^*) = \max_{\mathbf{x}_M} \sum_{\mathbf{x}_S} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$

- **NP-hard**: exponentially many terms
- We will focus on **approximation** algorithms
  - **Anytime**: very fast & very approximate ! Slower & more accurate

Monte Carlo estimators

- Most basic form: empirical estimate of probability

$$\mathbb{E}[u(x)] = \int p(x)u(x) \;\approx\; U = \frac{1}{m}\sum_i u(\tilde{x}^{(i)}) \qquad \tilde{x}^{(i)} \sim p(x)$$

- Relevant considerations
  - Able to sample from the target distribution p(x)?
  - Able to evaluate p(x) explicitly, or only up to a constant?

- "Any-time" properties $\mathbb{E}[U] = \mathbb{E}[u(x)]$
  - Unbiased estimator,

    or asymptotically unbiased, $\mathbb{E}[U] \to \mathbb{E}[u(x)]$ as $m \to \infty$

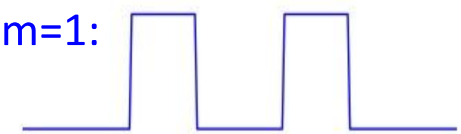
  - Variance of the estimator decreases with m

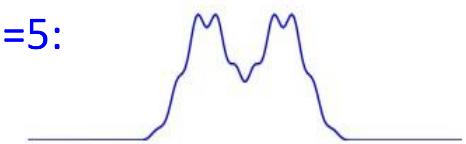
Monte Carlo estimators

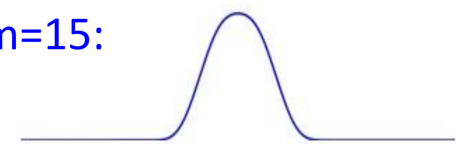
- Most basic form: empirical estimate of probability

$$\mathbb{E}[u(x)] = \int p(x)u(x) \;\approx\; U = \frac{1}{m}\sum_i u(\tilde{x}^{(i)}) \qquad \tilde{x}^{(i)} \sim p(x)$$

- Central limit theorem
  - p(U) is asymptotically Gaussian:

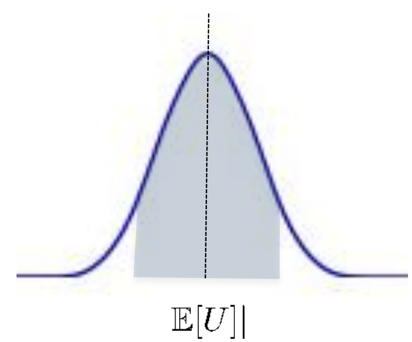
m=1: 

m=5: 

m=15: 

- Finite sample confidence intervals
  - If u(x) or its variance are bounded, e.g., $u(x^{(i)}) \in [0,1]$
    probability concentrates rapidly around the expectation:

$$\Pr\big[\, |U - \mathbb{E}[U]| > \epsilon \,\big] \le O\big(\, \exp(-m\epsilon^2)\,\big)$$



Estimating an Expectation: Monte Carlo Simulation

Since the sample mean is a function of the sample space, it has its own expectation and variance

Let $\mathrm{Av}_n(f)$ be a sample mean, where the function f has expectation μ and variance σ^2. The expectation of the sample mean $\mathrm{Av}_n(f)$ is μ and its variance is σ^2/n.

The estimate $\mathrm{Av}_n(f)$ is said to be unbiased

since the expectation of the estimate equals the quantity we are trying to estimate.

The variance of this estimate is inversely proportional to the sample size n

Estimating an Expectation: Monte Carlo Simulation

Central Limit Theorem

Let $\mathrm{Av}_n(f)$ be a sample mean, where the function f has expectation μ and variance σ^2. As the sample size n tends to infinity, the distribution of $\sqrt{n}(\mathrm{Av}_n(f) - \mu)$ converges to a Normal with mean 0 and variance σ^2. We say in this case that the estimate $\mathrm{Av}_n(f)$ is asymptotically Normal.

Continues to hold if we replace σ^2 by the sample variance:

$$\mathrm{S}^2{}_n(f) \stackrel{def}{=} \frac{1}{n-1} \sum_{i=1}^{n} \left(f(\mathbf{x}^i) - \mathrm{Av}_n(f)\right)^2$$

Allows us to compute confidence intervals, even when we do not know the value of variance σ^2.

A Sample

- Given a set of variables $X=\{X_1,\ldots,X_n\}$, a sample, denoted by S^t is an instantiation of all variables:

$$S^t = (x_1^t, x_2^t, \ldots, x_n^t)$$

How to Draw a Sample ? Univariate Distribution

- Example: Given random variable X having domain {0, 1} and a distribution $P(X) = (0.3, 0.7)$.

- Task: Generate samples of X from P.
- How?
  - draw random number $r \in [0, 1]$
  - If $(r < 0.3)$ then set $X=0$
  - Else set $X=1$

So, how to draw a sample from a multi-dimensional distribution?

How to Draw a Sample? Multi-Variate Distribution

- Let $X=\{X_1,..,X_n\}$ be a set of variables
- Express the distribution in product form

$$P(X) = P(X_1) \times P(X_2 \mid X_1) \times \ldots \times P(X_n \mid X_1, \ldots, X_{n-1})$$

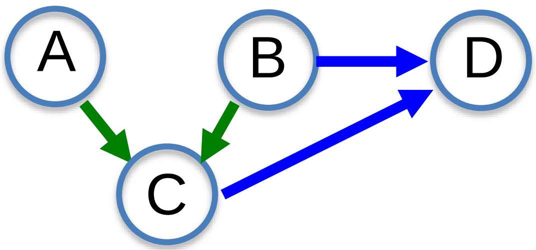
- Sample variables one by one from left to right, along the ordering dictated by the product form.
- Bayesian network literature: *Logic sampling* or *Forward Sampling.*

[e.g., Henrion 1988]

Sampling in Bayes nets (Forward Sampling)

- No evidence: "causal" form makes sampling easy
  - Follow variable ordering defined by parents
  - Starting from root(s), sample downward
  - When sampling each variable, condition on values of parents

$$p(A,B,C,D) = p(A)\; p(B)\; p(C \mid A,B)\; p(D \mid B,C)$$



Sample:

$$a \sim p(A)$$

$$b \sim p(B)$$

$$c \sim p(C \mid A=a, B=b)$$

$$d \sim p(D \mid C=c, B=b)$$

Froward Sampling: No Evidence (Henrion 1988)

Input: Bayesian network

$X = \{X_1, \ldots, X_N\}$, N- #nodes, T - # samples

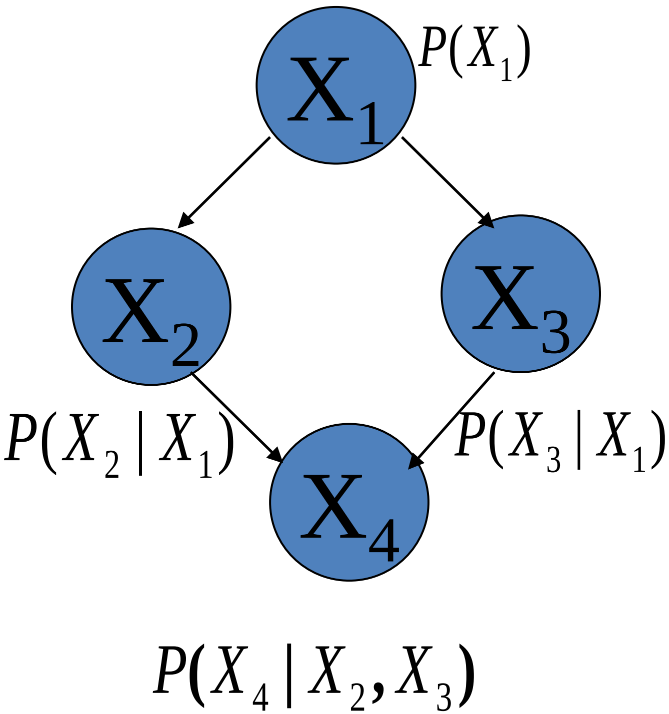
Output: T samples

Process nodes in topological order – first process the ancestors of a node, then the node itself:

1. For t = 0 to T
2. For i = 0 to N
3. $X_i \leftarrow$ sample x_i^t from $P(x_i \mid pa_i)$

Forward Sampling (example)

$$P(X_1, X_2, X_3, X_4) = P(X_1) \times P(X_2 \mid X_1) \times P(X_3 \mid X_1) \times P(X_4 \mid X_2, X_3)$$



No Evidence

// generate sample k

1. Sample x_1 from $P(x_1)$
2. Sample x_2 from $P(x_2 \mid X_1 = x_1)$
3. Sample x_3 from $P(x_3 \mid X_1 = x_1)$
4. Sample x_4 from $P(x_4 \mid X_2 = x_2, X_3 = x_3)$

No evidence!

Forward Sampling w/ Evidence

Input: Bayesian network

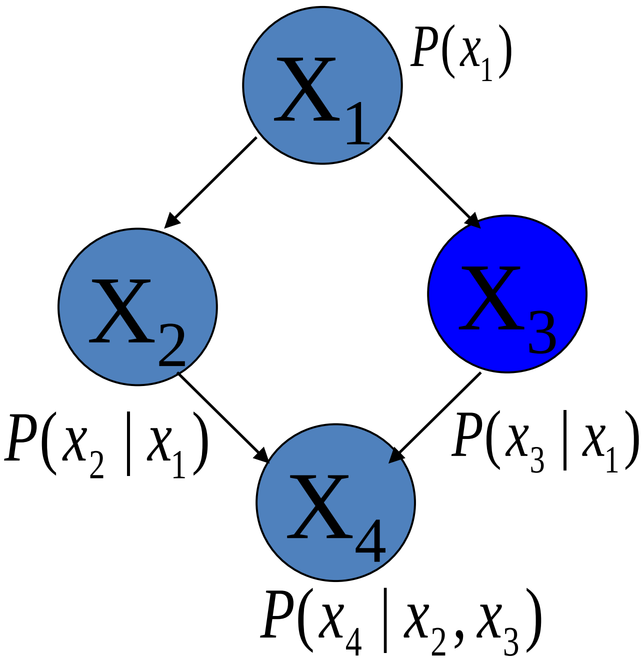
$X = \{X_1,\ldots,X_N\}$, N- #nodes

E – evidence, T - # samples

Output: T samples consistent with E

1. For t=1 to T
2. For i=1 to N
3. $X_i \leftarrow$ sample x_i^t from $P(x_i \mid pa_i)$
4. If X_i in E and $X_i \neq x_i$, reject sample:
5. Goto Step 1.

Forward Sampling (example)



$\text{Evidence}: X_3 = 0$

// generate sample k

1. Sample x_1 from $P(x_1)$
2. Sample x_2 from $P(x_2 \mid x_1)$
3. Sample x_3 from $P(x_3 \mid x_1)$
4. If $x_3 \neq 0$, reject sample and start from 1, otherwise
5. Sample x_4 from $P(x_4 \mid x_{2,} x_3)$

How to answer queries with sampling? Expected value and Variance

Many queries can be phrased as computing expectation of some functions

Expected value: Given a probability distribution P(X) and a function g(X) defined over a set of variables X = $\{X_1, X_2, \ldots X_n\}$, the expected value of g w.r.t. P is

$$E_P[g(x)] = \sum_x g(x)P(x)$$

Variance: The variance of g w.r.t. P is:

$$Var_P[g(x)] = \sum_x [g(x) - E_P[g(x)]]^2 P(x)$$

Monte Carlo Estimate

- **Estimator:**
  - An estimator is a function of the samples.
  - It produces an estimate of the unknown parameter of the sampling *distribution.*

Given i.i.d. samples $S^1, S^2, \ldots S^T$ drawn from P,

the Monte carlo estimate of $E_P[g(x)]$ is given by :

$$\hat{g} = \frac{1}{T}\sum_{t=1}^{T} g(S^t)$$

Example: Monte Carlo estimate

- Given:
  - A distribution P(X) = (0.3, 0.7).
  - g(X) = 40 if X equals 0
    = 50 if X equals 1.
- Estimate $E_P[g(x)]=(40x0.3+50x0.7)=47$.
- Generate k samples from P: 0,1,1,1,0,1,1,0,1,0

$$\hat{g} = \frac{40 \times \# samples(X=0) + 50 \times \# samples(X=1)}{\# samples}$$

$$= \frac{40 \times 4 + 50 \times 6}{10} = 46$$

Bayes Networks with Evidence

- Estimating posterior probabilities, P[A = a | E=e]?

- Rejection sampling
  - Draw x ~ p(x), but discard if E $\neq$ e
  - Resulting samples are from p(x | E=e); use as before
  - Problem: keeps only P[E=e] fraction of the samples!
  - Performs poorly when evidence probability is small

- Estimate the ratio: P[A=a,E=e] / P[E=e]
  - Two estimates (numerator & denominator)
  - Good finite **sample bounds** require low *relative* error!
  - Again, performs poorly when evidence probability is small
  - **What bounds can we get?**

Bayes Networks With Evidence

- Estimating the probability of evidence, P[E=e] (absolute error):

$$\mathrm{P}[E=e] = \mathbb{E}[\,\mathbb{1}[E=e]\,] \approx \; U = \frac{1}{m}\sum_i \mathbb{1}[\tilde{e}^{(i)} = e]$$

  - Finite sample bounds: u(x) ∈ [0,1] [e.g., Hoeffding]

$$\Pr\Big[|U - \mathbb{E}[U]| > \epsilon\Big] \le 2\exp(-2m\epsilon^2)$$

  - Relative error bounds [Dagum & Luby 1997]

$$\Pr\Big[\frac{|U - \mathbb{E}[U]|}{\mathbb{E}[U]} > \epsilon\Big] \le \delta \quad \text{if} \quad m \ge \frac{4}{\mathbb{E}[U]\epsilon^2}\log\frac{2}{\delta}$$

So, if U, the probability of evidence is very small we would need many many samples
That are not rejected.

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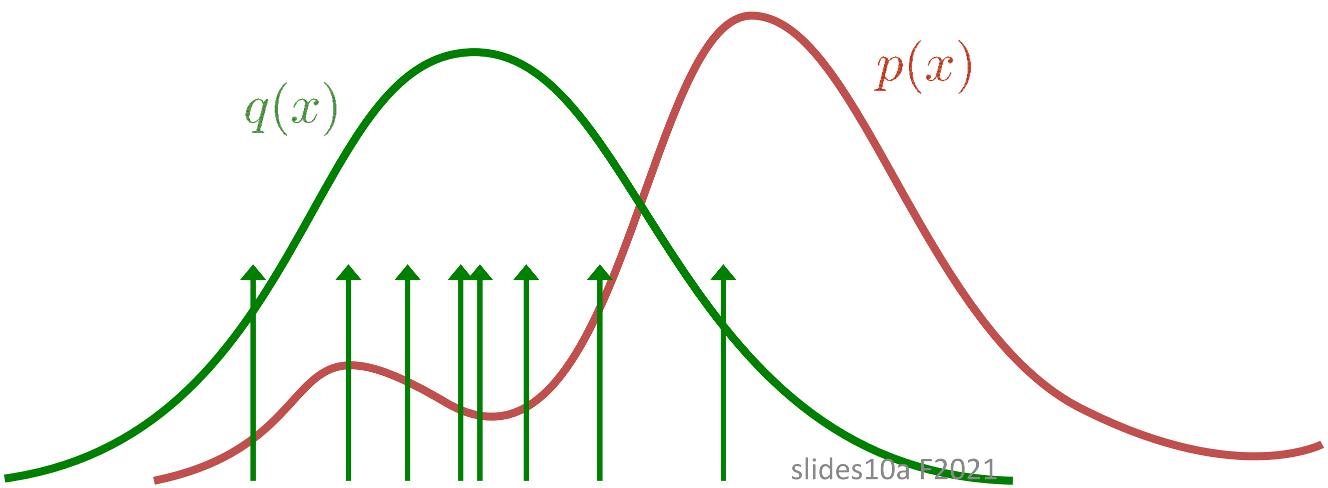
Importance Sampling: Main Idea

- Express query as the expected value of a random variable w.r.t. to a distribution Q.
- Generate random samples from Q.
- Estimate the expected value from the generated samples using a monte carlo estimator (average).

Importance Sampling

- $\mathbb{E}[u(x)] = \int p(x)u(x) \quad \approx \quad \hat{u} = \frac{1}{m}\sum_i u(\tilde{x}^{(i)}) \qquad \tilde{x}^{(i)} \sim p(x)$

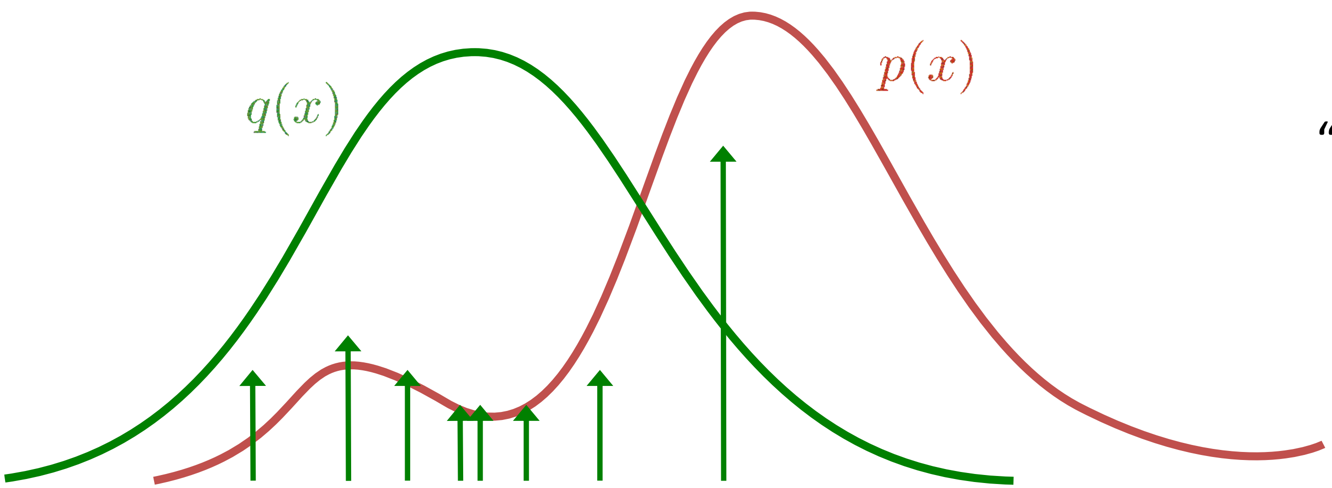
$$\int p(x)u(x) = \int q(x)\frac{p(x)}{q(x)}u(x) \approx \frac{1}{m}\sum_i \frac{p(\tilde{x}^{(i)})}{q(\tilde{x}^{(i)})}u(\tilde{x}^{(i)}) \qquad \tilde{x}^{(i)} \sim q(x)$$



Importance Sampling

- $\mathbb{E}[u(x)] = \int p(x)u(x) \approx \hat{u} = \frac{1}{m}\sum_i u(\tilde{x}^{(i)}) \qquad \tilde{x}^{(i)} \sim p(x)$

$$\int p(x)u(x) = \int q(x)\frac{p(x)}{q(x)}u(x) \approx \frac{1}{m}\sum_i \frac{p(\tilde{x}^{(i)})}{q(\tilde{x}^{(i)})}u(\tilde{x}^{(i)}) \qquad \tilde{x}^{(i)} \sim q(x)$$



"importance weights"

$$w^{(i)} = \frac{p(\tilde{x}^{(i)})}{q(\tilde{x}^{(i)})}$$

Estimating P(E) and P(X|e)

Importance Sampling For P(e)
(for discrete variables)

$Let\ Z = X \setminus E,$

Let Q(Z) be a (proposal) distribution, satisfying

$P(z,e) > 0 \Rightarrow Q(z) > 0$

Then, we can rewrite P(e) as :

$$P(e) = \sum_{z} P(z,e) = \sum_{z} P(z,e) \frac{Q(z)}{Q(z)} = E_Q\left[\frac{P(z,e)}{Q(z)}\right] = E_Q[w(z)]$$

Monte Carlo estimate :

$$\hat{P}(e) = \frac{1}{T}\sum_{t=1}^{T} w(z^t), \text{where } z^t \leftarrow Q(Z)$$

Properties of IS Estimate of P(e)

- **Convergence:** by law of large numbers

$$\hat{P}(e) = \frac{1}{T}\sum_{i=1}^{T} w(z^i) \xrightarrow{a.s.} P(e) \text{ for T} \rightarrow \infty$$

- **Unbiased.**

$$E_Q[\hat{P}(e)] = P(e)$$

- **Variance:**

$$Var_Q\left[\hat{P}(e)\right] = Var_Q\left[\frac{1}{T}\sum_{i=1}^{N} w(z^i)\right] = \frac{Var_Q[w(z)]}{T}$$

Properties of IS Estimate of P(e)

- Mean Squared Error of the estimator

$$MSE_Q\left[\hat{P}(e)\right] = E_Q\left[\left(\hat{P}(e) - P(e)\right)^2\right]$$

$$= \left(P(e) - E_Q[\hat{P}(e)]\right)^2 + Var_Q\left[\hat{P}(e)\right]$$

$$= Var_Q\left[\hat{P}(e)\right]$$

$$= \frac{Var_Q[w(x)]}{T}$$

This quantity enclosed in the brackets is zero because the expected value of the estimator equals the expected value of g(x)

Estimating $P(X_i|e)$

Let $\delta_{x_i}(z)$ be a dirac - delta function, which is 1 if z contains x_i and 0 otherwise.

$$P(x_i \mid e) = \frac{P(x_i, e)}{P(e)} = \frac{\sum_z \delta_{x_i}(z) P(z,e)}{\sum_z P(z,e)} = \frac{E_Q\left[\frac{\delta_{x_i}(z)P(z,e)}{Q(z)}\right]}{E_Q\left[\frac{P(z,e)}{Q(z)}\right]}$$

Idea : Estimate numerator and denominator by IS.

$$\text{Ratio estimate : } \overline{P}(x_i \mid e) = \frac{\hat{P}(x_i, e)}{\hat{P}(e)} = \frac{\sum_{k=1}^{T} \delta_{x_i}(z^k) w(z^k, e)}{\sum_{k=1}^{T} w(z^k, e)}$$

$$\text{Estimate is biased : } \mathrm{E}\left[\overline{P}(x_i \mid e)\right] \neq P(x_i \mid e)$$

Properties of the IS estimator for $P(X_i|e)$

- Convergence: By Weak law of large numbers

$$\overline{P}(x_i \mid e) \to P(x_i \mid e) \text{ as T} \to \infty$$

- Asymptotically unbiased

$$\lim_{T\to\infty} E_P[\overline{P}(x_i \mid e)] = P(x_i \mid e)$$

- Variance
  - Harder to analyze
  - Liu suggests a measure called "Effective sample size"

End of class

Generating Samples From Q

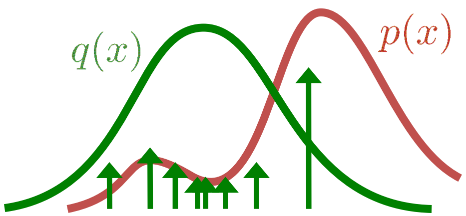
- No restrictions on “how to”
- Typically, express Q in product form:
  - $Q(Z)=Q(Z_1)xQ(Z_2|Z_1)x....xQ(Z_n|Z_1,..Z_{n-1})$
- Sample along the order $Z_1,..,Z_n$
- Example:
  - $Z_1 \leftarrow Q(Z_1)=(0.2,0.8)$
  - $Z_2 \leftarrow Q(Z_2|Z_1)=(0.1,0.9,0.2,0.8)$
  - $Z_3 \leftarrow Q(Z_3|Z_1,Z_2)=Q(Z_3)=(0.5,0.5)$

More on Properties of IS

- Importance sampling:

$$\int p(x)u(x) = \int q(x)\frac{p(x)}{q(x)}u(x) \approx \frac{1}{m}\sum_i \frac{p(\tilde{x}^{(i)})}{q(\tilde{x}^{(i)})}u(\tilde{x}^{(i)}) \qquad \tilde{x}^{(i)} \sim q(x)$$

- IS is unbiased and fast if q(.) is easy to sample from
- IS can have lower variance if q(.) is chosen well
  - Ex: q(x) puts more probability mass where u(x) is large
  - Optimal: q(x) ∝ |u(x) p(x)|
- IS can also give poor performance
  - If q(x) << u(x) p(x): rare but very high weights!
  - Then, empirical variance is also unreliable!
  - For guarantees, need to analytically bound weights / variance…



How to get a good proposal?

Likelihood Weighting

(Fung and Chang, 1990; Shachter and Peot, 1990)

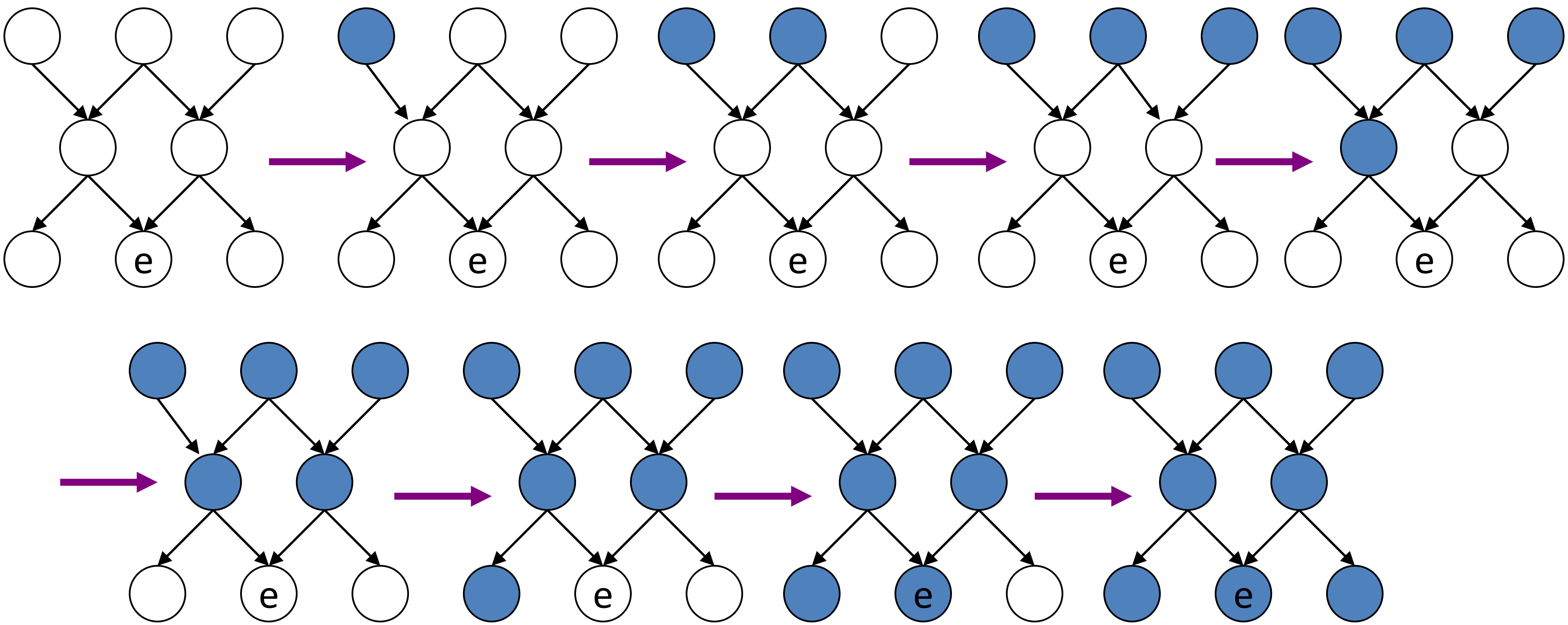
Is an instance of importance sampling!

"Clamping" evidence+
logic sampling (Forward sampling)+
weighing samples by evidence likelihood

Works well for *likely evidence!*

Likelihood Weighting: Sampling

Sample in topological order over **X** !



Clamp evidence, Sample $x_i \leftarrow P(X_i|pa_i)$, $P(X_i|pa_i)$ is *a look-up* in CPT!

Likelihood Weighting: Proposal Distribution

$$Q(X \setminus E) = \prod_{X_i \in X \setminus E} P(X_i \mid pa_i, e)$$

Notice: Q is another Bayesian network

Example :

Given a Bayesian network : $\mathrm{P}(\mathrm{X}_1, \mathrm{X}_2, \mathrm{X}_3) = \mathrm{P}(\mathrm{X}_1) \times \mathrm{P}(\mathrm{X}_2 \mid \mathrm{X}_1) \times \mathrm{P}(\mathrm{X}_3 \mid \mathrm{X}_1, \mathrm{X}_2)$ and

Evidence $\mathrm{X}_2 = \mathrm{x}_2$.

$$\mathrm{Q}(\mathrm{X}_1, \mathrm{X}_3) = P(X_1) \times P(X_3 \mid X_1, X_2 = x_2)$$

Weights :

Given a sample : $x = (x_1, .., x_n)$

$$w = \frac{P(x,e)}{Q(x)} = \frac{\cancel{\prod_{X_i \in X \setminus E} P(x_i \mid pa_i, e)} \times \prod_{E_j \in E} P(e_j \mid pa_j)}{\cancel{\prod_{X_i \in X \setminus E} P(x_i \mid pa_i, e)}}$$

$$= \prod_{E_j \in E} P(e_j \mid pa_j)$$

Likelihood Weighting: Estimates

Estimate P(e):

$$\hat{P}(e) = \frac{1}{T}\sum_{t=1}^{T} w^{(t)}$$

Estimate Posterior Marginals:

$$\hat{P}(x_i \mid e) = \frac{\hat{P}(x_i, e)}{\hat{P}(e)} = \frac{\sum_{t=1}^{T} w^{(t)} g_{x_i}(x^{(t)})}{\sum_{t=1}^{T} w^{(t)}}$$

$$g_{x_i}(x^{(t)}) = 1 \text{ if } \mathrm{x_i} = x_i^t \text{ and equals zero otherwise}$$

Properties of Likelihood Weighting

- Converges to exact posterior marginals
- Generates Samples Fast
- Sampling distribution is close to prior (especially if $E \subset$ Leaf Nodes)
- Increasing sampling variance

$\Rightarrow$Convergence may be slow

$\Rightarrow$Many samples with $P(x^{(t)})=0$ rejected

(because weight is zero)

Outline

- Definitions and Background on Statistics
- Theory of importance sampling
- Likelihood weighting
- **State-of-the-art importance sampling techniques**

Proposal selection

- One should try to select a proposal that is as close as possible to the posterior distribution.

$$Var_Q\left[\hat{P}(e)\right] = \frac{Var_Q[w(z)]}{T} = \frac{1}{N}\sum_{z \in Z}\left(\frac{P(z,e)}{Q(z)} - P(e)\right)^2 Q(z)$$

$$\frac{P(z,e)}{Q(z)} - P(e) = 0, \text{ to have a zero-variance estimator}$$

$$\therefore \frac{P(z,e)}{P(e)} = Q(z)$$

$$\therefore Q(z) = P(z \mid e)$$

Perfect sampling using Bucket Elimination

- Algorithm:
  - Run Bucket elimination on the problem along an ordering $d=(X_N,..,X_1)$.
  - Sample along the reverse ordering: $(X_1,..,X_N)$
  - At each variable X_i, recover the probability $P(X_i|x_1,...,x_{i-1})$ by referring to the bucket.

How to sample from a Markov network? Exact sampling via inference

- Draw samples from P[A|E=e] directly?
  - Model defines un-normalized p(A,…,E=e)
  - Build (oriented) tree decomposition & sample

$$\tilde{\mathbf{b}} \sim f(\tilde{a}, b) \cdot f(b, \tilde{c}) \cdot f(b, \tilde{d}) \cdot f(b, \tilde{e}) / \lambda_{B \to C}$$

$$\tilde{\mathbf{c}} \sim f(c, \tilde{a}) \cdot f(c, \tilde{e}) \cdot \lambda_{B \to C}(\tilde{a}, c, \tilde{d}, \tilde{e}) / \lambda_{C \to D}$$

$$\tilde{\mathbf{d}} \sim f(\tilde{a}, d) \cdot \lambda_{B \to D}(d, \tilde{e}) / \lambda_{D \to E}(\tilde{a}, \tilde{e})$$

$$\tilde{\mathbf{e}} \sim \lambda_{D \to E}(\tilde{a}, e) / \lambda_{E \to A}(\tilde{a})$$

$$\tilde{\mathbf{a}} \sim p(A) = f(a) \cdot \lambda_{E \to A}(a) / Z$$

B: $\overbrace{f(a,b)\ \ f(b,c)\ \ f(b,d)\ \ f(b,e)}^{\sum_b \prod}$

C: $f(c,a)\ \ f(c,e)\ \ \lambda_{B \to C}(a,c,d,e)$

D: $f(a,d)\ \ \lambda_{C \to D}(d,e,a)$

E: $\lambda_{D \to E}(a,e)$

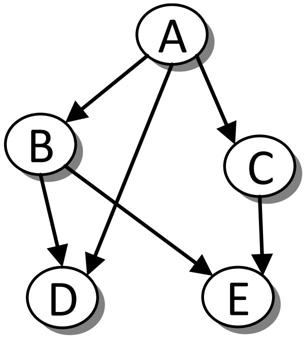
A: $f(a)\ \ \lambda_{E \to A}(a)$

Z

Downward message normalizes bucket; ratio is a conditional distribution

Work: O(exp(w)) to build distribution
O(n d) to draw each sample

Bucket elimination (BE)



$\sum_b \prod$ Elimination operator

bucket B: $\underbrace{P(B|A) \quad P(D|B,A) \quad P(e|B,C)}$

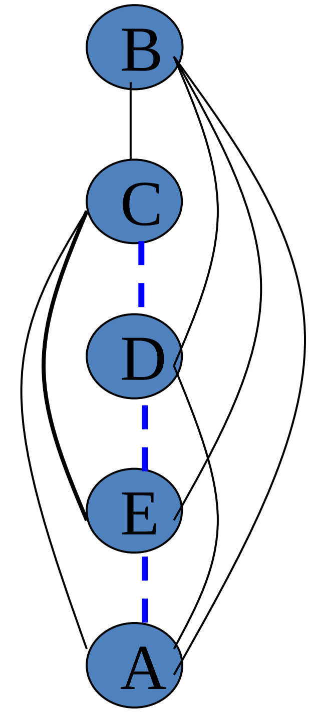
bucket C: $P(C|A)$ $\boldsymbol{h^B(A,D,C,e)}$

bucket D: $\boldsymbol{h^C(A,D,e)}$

bucket E: $\boldsymbol{h^D(A,e)}$

bucket A: $P(a)$ $\boldsymbol{h^E(a)}$

$\boldsymbol{P(e)}$



Sampling from the output of BE

(Dechter 2002)

$\text{Set } A = a, D = d, C = c \text{ in the bucket}$

$\text{Sample}: B = b \leftarrow Q(B \mid a, e, d) \propto P(B \mid a) P(d \mid B, a) P(e \mid b, c)$

bucket B: P(B|A) P(D|B,A) P(e|B,C)

bucket C: P(C|A) $h^B(A,D,C,e)$

$\text{Set } A = a, D = d \text{ in the bucket}$

$\text{Sample}: C = c \leftarrow Q(C \mid a, e, d) \propto P(C \mid A) \cdot h^B(a, d, C, e$

bucket D: $h^C(A,D,e)$

$\text{Set } A = a \text{ in the bucket}$

$\text{Sample}: D = d \leftarrow Q(D \mid a, e) \propto h^C(a, D, e)$

bucket E: $h^D(A,e)$

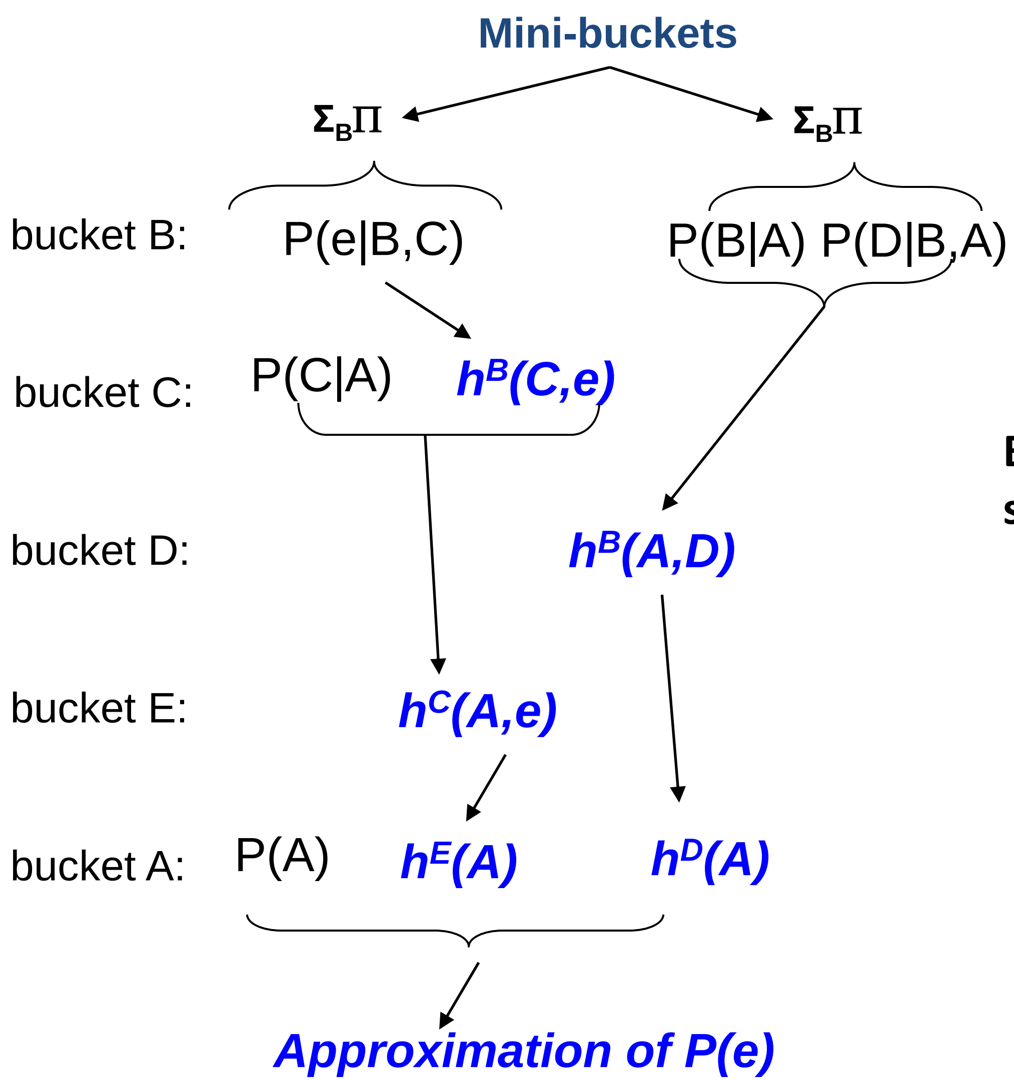
$\text{Evidence bucket}: \text{ignore}$

bucket A: P(A) $h^E(A)$

$$\mathbf{Q(A) \propto P(A) \times h^E(A)}$$

$$\mathit{Sample}: \mathbf{A = a \leftarrow Q(A)}$$

Mini-Bucket Elimination



Space and Time constraints: Maximum scope size of the new function generated should be bounded by 2

BE generates a function having scope size 3. So it cannot be used.

Sampling from the output of MBE

bucket B: P(e|B,C) | P(B|A) P(D|B,A)

bucket C: P(C|A) $h^B(C,e)$

bucket D: $h^B(A,D)$

bucket E: $h^C(A,e)$

bucket A: $h^E(A)$ $h^D(A)$

Sampling is same as in BE-sampling except that now we construct Q from a randomly selected "mini-bucket"

IJGP-Sampling
(Gogate and Dechter, 2005)

- Iterative Join Graph Propagation (IJGP)
  - A Generalized Belief Propagation scheme (Yedidia et al., 2002)
- IJGP yields better approximations of P(X|E) than MBE (Dechter, Kask and Mateescu, 2002)
- Output of IJGP is same as mini-bucket "clusters"
- **Currently one of the best performing IS scheme!**

[Liu, Fisher, Ihler 2015]

Choosing a proposal (wmb-IS)

- Can use WMB upper bound to define a proposal q(x):

$$\tilde{\mathbf{b}} \sim w_1\, q_1(b|\tilde{a},\tilde{c}) \;+\; w_2\, q_2(b|\tilde{d},\tilde{e})$$

Weighted mixture:

use minibucket 1 with probability w_1

or, minibucket 2 with probability $w_2 = 1 - w_1$

where

$$q_1(b|a,c) = \left[\frac{f(a,b)\cdot f(b,c)}{\lambda_{B\to C}(a,c)}\right]^{\frac{1}{w_1}}$$

⋮

$$\tilde{\mathbf{a}} \sim q(A) = f(a)\cdot \lambda_{E\to A}(a)/U$$

mini-buckets

B: $\underbrace{\overbrace{f(a,b)\;\; f(b,c)}^{w_1}}$ $\quad$ $\underbrace{\overbrace{f(b,d)\;\; f(b,e)}^{w_2}}$

C: $\underbrace{f(c,a)\;\; f(c,e)\;\; \lambda_{B\to C}(a,c)}$

D: $\underbrace{f(a,d)\;\; \lambda_{B\to D}(d,e)}$

E: $\underbrace{\lambda_{C\to E}(a,e)\;\; \lambda_{E\to E}(a,e)}$

A: $\underbrace{f(a)\;\; \lambda_{E\to A}(a)}$

U = upper bound

Key insight: provides bounded importance weights!

$$0 \le \frac{F(x)}{q(x)} \le U \qquad\qquad \forall x$$

WMB-IS Bounds [Liu, Fisher, Ihler 2015]

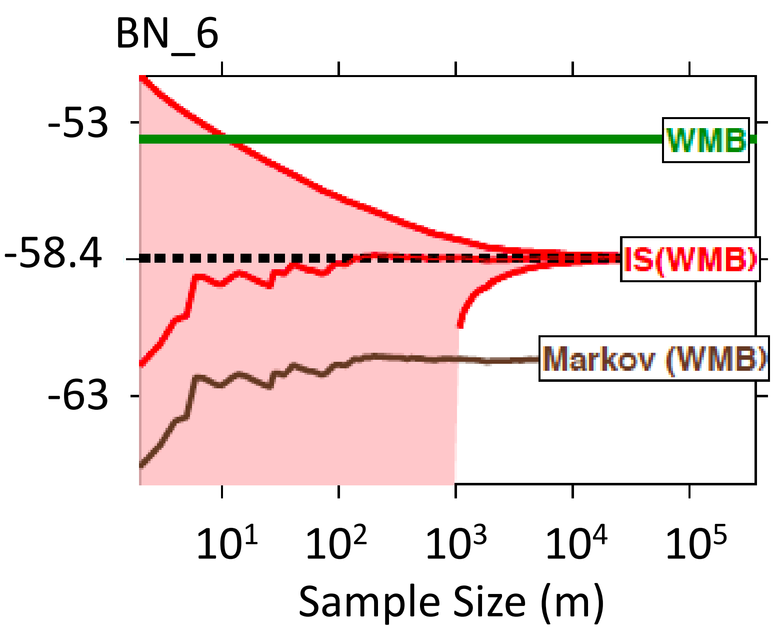
- Finite sample bounds on the average

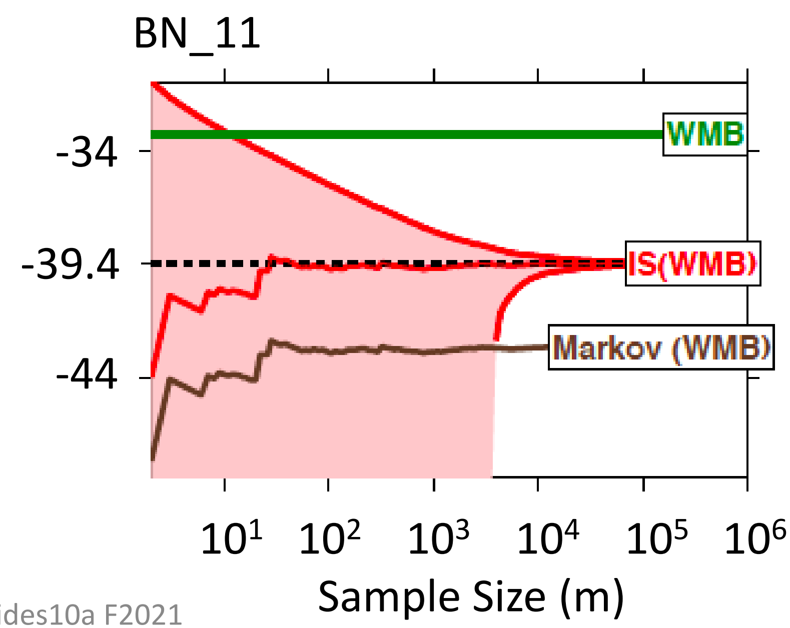
$$\Pr\left[|\hat{Z} - Z| > \epsilon\right] \leq 1 - \delta$$

$$\epsilon = \sqrt{\frac{2\hat{V}\log(4/\delta)}{m}} + \frac{7\,U\,\log(4/\delta)}{3(m-1)}$$

"Empirical Bernstein" bounds

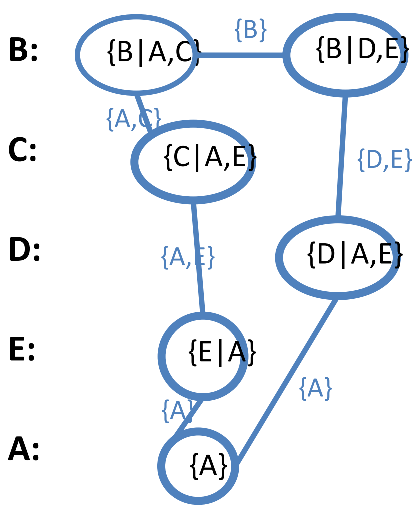
- Compare to forward sampling





Other Choices of Proposals

- Belief propagation
  - BP-based proposal [Changhe & Druzdzel 2003]
  - Join-graph BP proposal [Gogate & Dechter 2005]
  - Mean field proposal [Wexler & Geiger 2007]



- Adaptive importance sampling
  - Use already-drawn samples to update q(x)
  - Ex:

    [Cheng & Druzdzel 2000]

    [Lapeyre & Boyd 2010]

    …
  - Lose “iid”-ness of samples

Overview

1. Probabilistic Reasoning/Graphical models
2. Importance Sampling
3. Markov Chain Monte Carlo: Gibbs Sampling
4. Sampling in presence of Determinism
5. Rao-Blackwellisation
6. AND/OR importance sampling