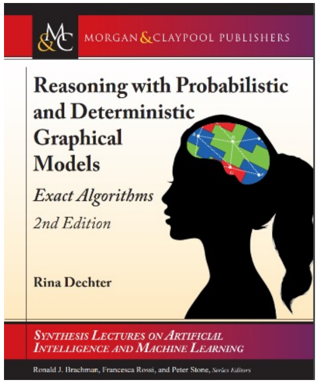
Reasoning with Graphical Models

Slides Set 5:

Exact Inference Algorithms Bucket-elimination

Rina Dechter



(Dechter chapter 4, Darwiche chapter 6)

Inference for probabilistic networks

- Bucket elimination (Dechter chapter 4)
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
  - Influence diagrams ?
- Induced-Width (Dechter, Chapter 3.4)

Inference for probabilistic networks

- Bucket elimination
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
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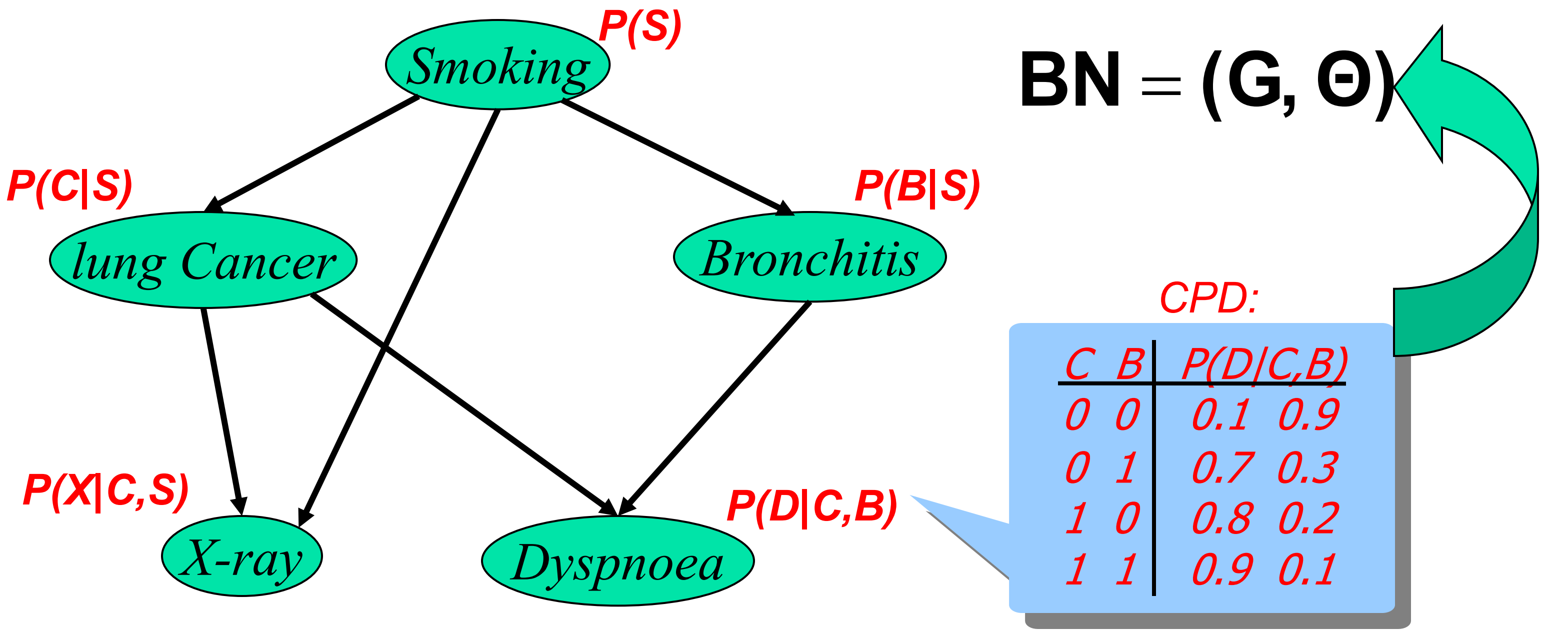
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width

Bayesian Networks: Example
(Pearl, 1988)

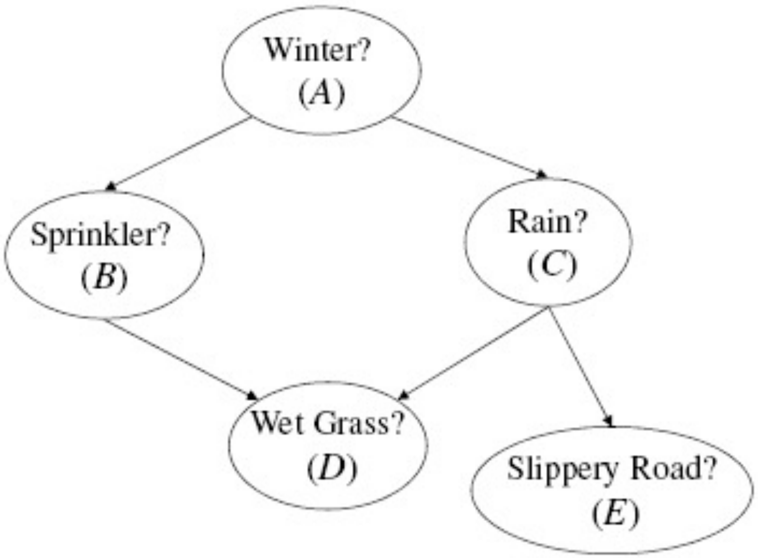


$$P(S, C, B, X, D) = P(S)\,P(C|S)\,P(B|S)\,P(X|C,S)\,P(D|C,B)$$

Belief Updating:

P (lung cancer=yes | smoking=no, dyspnoea=yes) = ?

A Bayesian Network



A	Θ_A
true	.6
false	.4

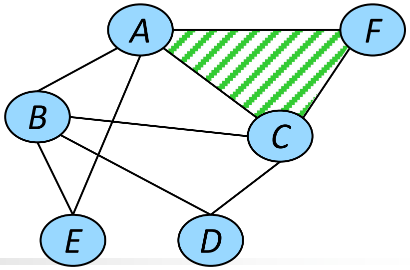
A	B	$\Theta_{B\|A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25

A	C	$\Theta_{C\|A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

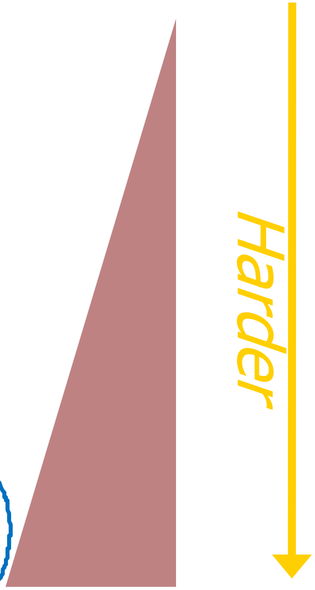
B	C	D	$\Theta_{D\|BC}$
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

C	E	$\Theta_{E\|C}$
true	true	.7
true	false	.3
false	true	0
false	false	1

Types of queries



Max-Inference	$f(\mathbf{x}^*) = \max_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Sum-Inference	$Z = \sum_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Mixed-Inference	$f(\mathbf{x}_M^*) = \max_{\mathbf{x}_M} \sum_{\mathbf{x}_S} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$



- **NP-hard**: exponentially many terms
- We will focus on exact and then on **approximation** algorithms
  - **Anytime**: very fast & very approximate ! Slower & more accurate

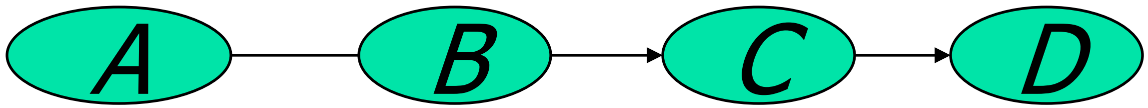
Belief Updating is NP-hard

- Each SAT formula can be mapped into a belief updating query in a Bayesian network
- Example

$$(\neg u \vee \neg w \vee y) \wedge (u \vee \neg v \vee w)$$

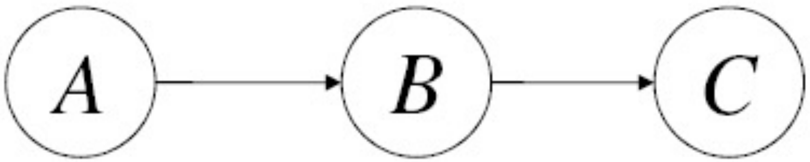
A Simple Network

Given:



- How can we compute P(D)?, P(D|A=0)? P(A|D=0)?
- Brute force O(k^4)
- Maybe O($4k^2$)

Elimination as a Basis for Inference



A	Θ_A
true	.6
false	.4

A	B	$\Theta_{B\|A}$
true	true	.9
true	false	.1
false	true	.2
false	false	.8

B	C	$\Theta_{C\|B}$
true	true	.3
true	false	.7
false	true	.5
false	false	.5

To compute the prior marginal on variable C, $\Pr(C)$

we first eliminate variable A and then variable B

Elimination as a Basis for Inference

- There are two factors that mention variable A, Θ_A and $\Theta_{B|A}$
- We multiply these factors first and then sum out variable A from the resulting factor.
- Multiplying Θ_A and $\Theta_{B|A}$:

A	B	$\Theta_A \Theta_{B|A}$
true	true	.54
true	false	.06
false	true	.08
false	false	.32

- Summing out variable A:

B	$\sum_A \Theta_A \Theta_{B|A}$
true	$.62 = .54 + .08$
false	$.38 = .06 + .32$

Elimination as a Basis for Inference

- We now have two factors, $\sum_A \Theta_A \Theta_{B|A}$ and $\Theta_{C|B}$, and we want to eliminate variable B
- Since B appears in both factors, we must multiply them first and then sum out B from the result.
- Multiplying:

B	C	$\Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$
true	true	.186
true	false	.434
false	true	.190
false	false	.190

- Summing out:

C	$\sum_B \Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$
true	.376
false	.624

Elimination as a Basis for Inference

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- Since B appears in both factors, we must multiply them first and then sum out B from the result.
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B	C	$\Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$
true	true	.186
true	false	.434
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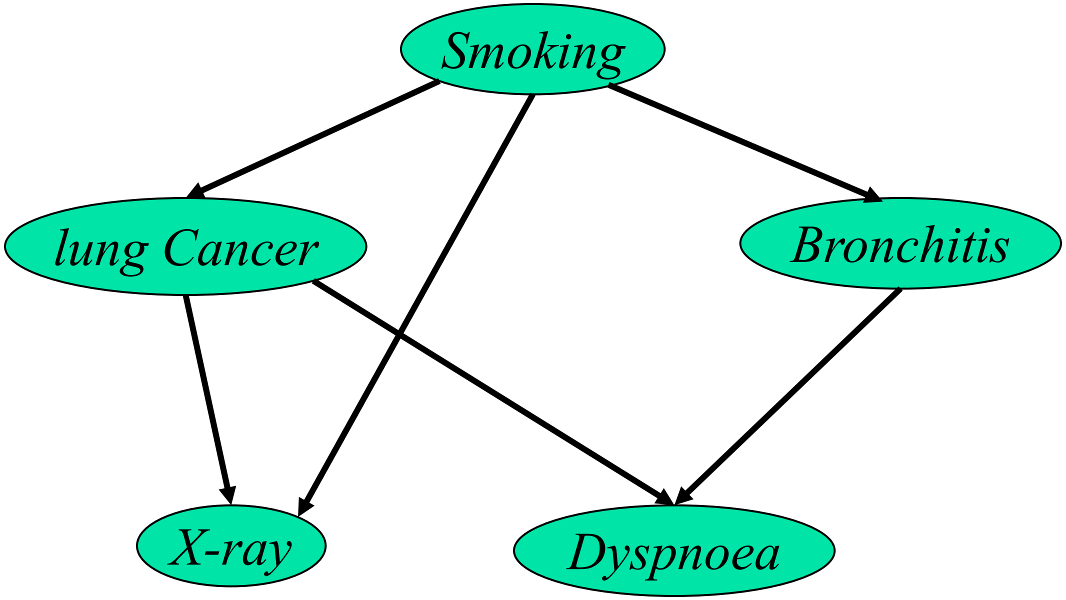
false	false	.190

- Summing out:

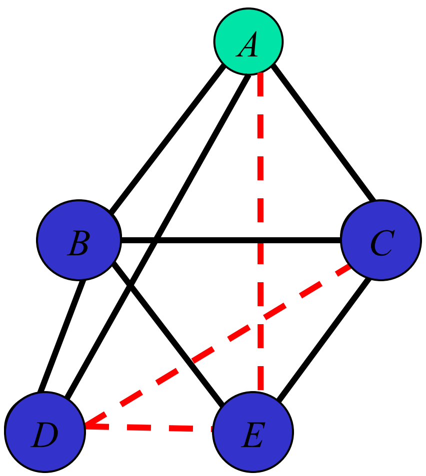
C	$\sum_B \Theta_{C\|B} \sum_A \Theta_A \Theta_{B\|A}$
true	.376
false	.624

Belief Updating



P (lung cancer=yes | smoking=no, dyspnoea=yes) = ?

Belief updating: P(X|evidence)=?



“Moral” graph

$$P(a|e=0) \propto P(a,e=0)=$$

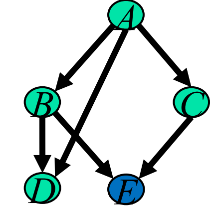
$$\sum_{e=0,d,c,b} P(a)\underbrace{P(b|a)}P(c|a)\underbrace{P(d|b,a)P(e|b,c)}=$$

$$P(a)\sum_{e=0}\sum_{d}\sum_{c}P(c|a\underbrace{\sum_{b}P(b|a)P(d|b,a)P(e|b,c)}_{h^{B}(a,d,c,e)}$$

Variable Elimination

Bucket elimination

Algorithm *BE-bel* (Dechter 1996)



$$P(A \mid E=0) = \alpha \sum_{E=0,D,C,B} P(A)\cdot P(B\mid A)\cdot P(C\mid A)\cdot P(D\mid A,B)\cdot P(E\mid B,C)$$

$\sum_b \prod$ ← *Elimination operator*

bucket B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

bucket C: $P(c|a)$ $\lambda^B(a,d,c,e)$

bucket D: $\lambda^C(a,d,e)$

bucket E: $e=0$ $\lambda^D(a,e)$

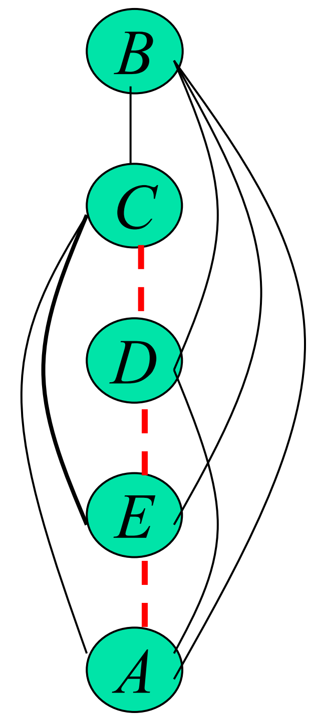
bucket A: $P(a)$ $\lambda^E(a)$

$P(e=0)$

$P(a,e=0)$

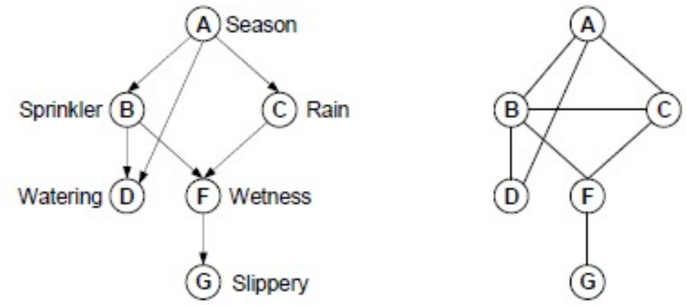
W=4*

"induced width" (max clique size)



$$P(a|e=0) = \frac{P(a,e=0)}{P(e=0)}$$

A Bayesian Network
Ordering: A,C,B,E,D,G



(a) Directed acyclic graph (b) Moral graph

$$P(a, g = 1) = \sum_{c,b,e,d,g=1} P(a, b, c, d, e, g) = \sum_{c,b,f,d,g=1} P(g|f)P(f|b, c)P(d|a, b)P(c|a)P(b|a)P(a).$$

$$P(a, g = 1) = P(a) \sum_{c} P(c|a) \sum_{b} P(b|a) \sum_{f} P(f|b, c) \sum_{d} P(d|b, a) \sum_{g=1} P(g|f). \quad (4.1)$$

$$P(a, g = 1) = P(a) \sum_{c} P(c|a) \sum_{b} P(b|a) \sum_{f} P(f|b, c)\lambda_G(f) \sum_{d} P(d|b, a). \quad (4.2)$$

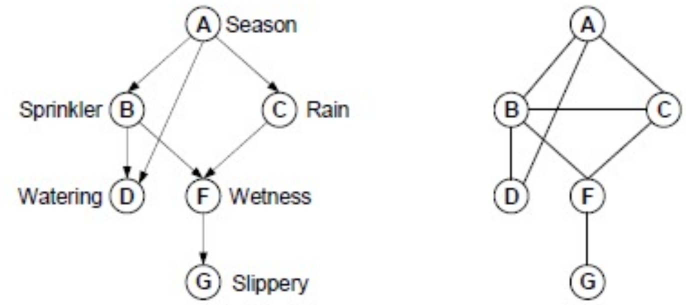
$$P(a, g = 1) = P(a) \sum_{c} P(c|a) \sum_{b} P(b|a)\lambda_D(a, b) \sum_{f} P(f|b, c)\lambda_G(f) \quad (4.3)$$

$$P(a, g = 1) = P(a) \sum_{c} P(c|a) \sum_{b} P(b|a)\lambda_D(a, b)\lambda_F(b, c) \quad (4.4)$$

$$P(a, g = 1) = P(a) \sum_{c} P(c|a)\lambda_B(a, c) \quad (4.5)$$

A Bayesian Network

Ordering: A,C,B,E,D,G



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$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\sum_{f} P(f|b,c)\sum_{d} P(d|b,a)\sum_{g=1} P(g|f). \quad (4.1)$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\sum_{f} P(f|b,c)\lambda_G(f)\sum_{d} P(d|b,a). \quad (4.2)$$

$$P(a, g=1) = P(a)\sum_{c} P(c|a)\sum_{b} P(b|a)\lambda_D(a,b)\sum_{f} P(f|b,c)\lambda_G(f) \quad (4.3)$$

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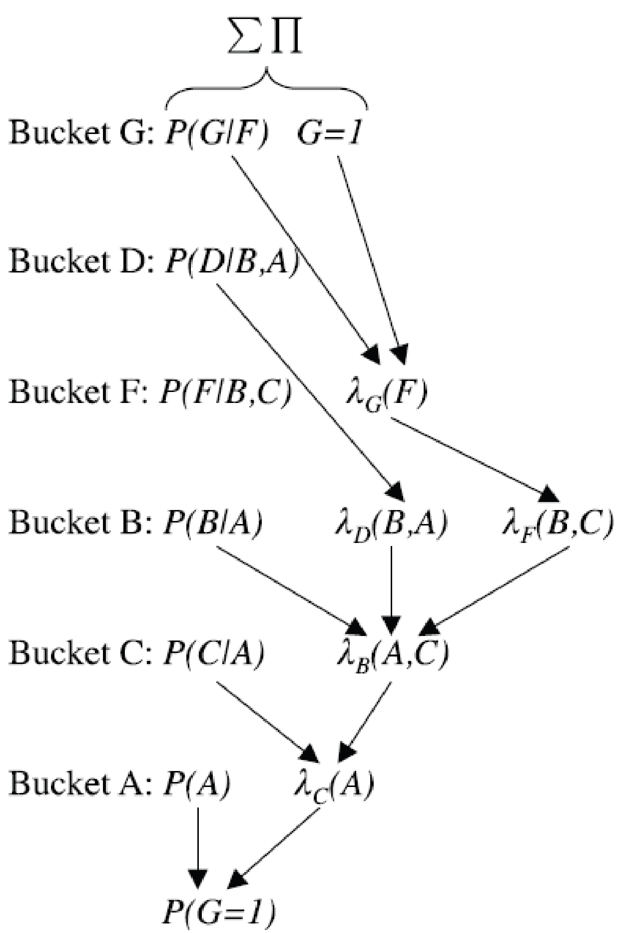
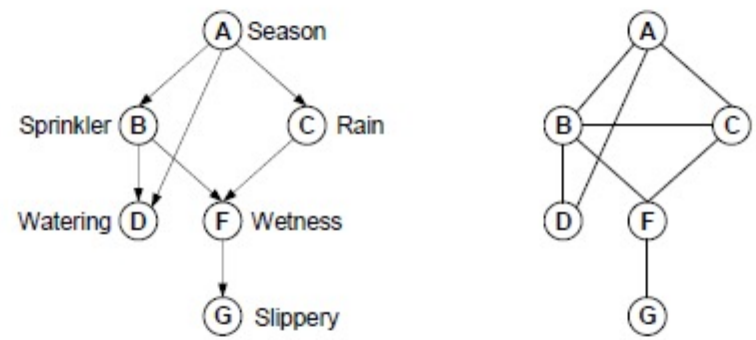
$$P(a, g=1) = P(a)\sum_{c} P(c|a)\lambda_B(a,c) \quad (4.5)$$

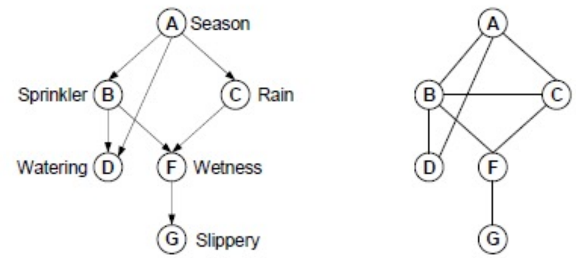
A Bayesian Network
Ordering: A,C,B,F,D,G





(a) Directed acyclic graph (b) Moral graph

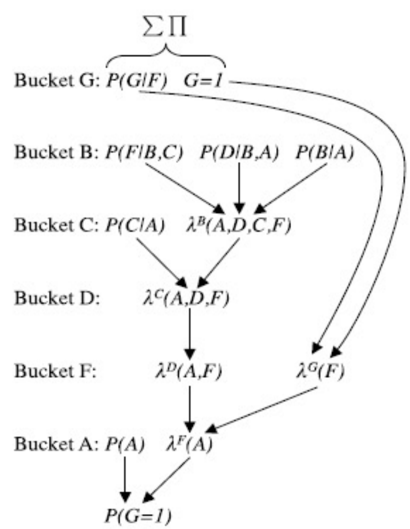
A Different Ordering



(a) Directed acyclic graph (b) Moral graph

Ordering: A,F,D,C,B,G

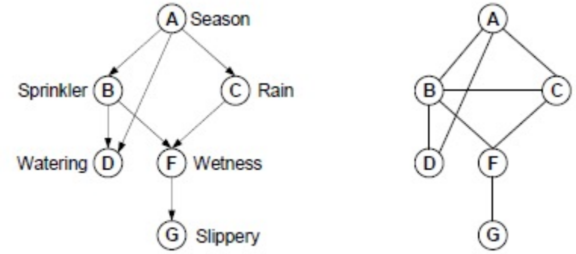
$$P(a, g=1) = P(a)\sum_f \sum_d \sum_c P(c|a) \sum_b P(b|a)\; P(d|a,b) P(f|b,c) \sum_{g=1} P(g|f)$$
$$= P(a)\sum_f \lambda_G(f) \sum_d \sum_c P(c|a) \sum_b P(b|a)\; P(d|a,b) P(f|b,c)$$
$$= P(a)\sum_f \lambda_G(f) \sum_d \sum_c P(c|a) \lambda_B(a,d,c,f)$$
$$= P(a)\sum_f \lambda_g(f) \sum_d \lambda_C(a,d,f)$$
$$= P(a)\sum_f \lambda_G(f) \lambda_D(a,f)$$
$$= P(a)\lambda_F(a)$$



(a)

Figure 4.3: The bucket's output when processing along $d_2 = A, F, D, C, B, G$

A Different Ordering



(a) Directed acyclic graph (b) Moral graph

Ordering: A,F,D,C,B,G

$$P(a, g=1) = P(a)\sum_f \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b)P(f|b,c) \sum_{g=1} P(g|f)$$
$$= P(a)\sum_f \lambda_G(f) \sum_d \sum_c P(c|a) \sum_b P(b|a)\ P(d|a,b)P(f|b,c)$$
$$= P(a)\sum_f \lambda_G(f) \sum_d \sum_c P(c|a)\lambda_B(a,d,c,f)$$
$$= P(a)\sum_f \lambda_g(f) \sum_d \lambda_C(a,d,f)$$
$$= P(a)\sum_f \lambda_G(f)\lambda_D(a,f)$$
$$= P(a)\lambda_F(a)$$

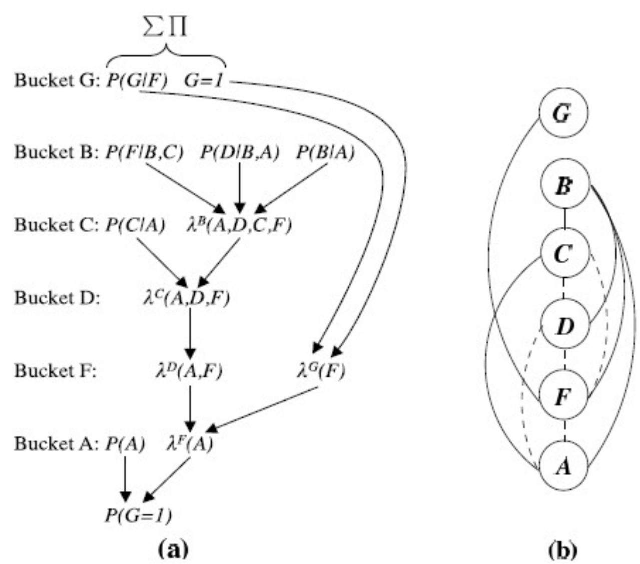
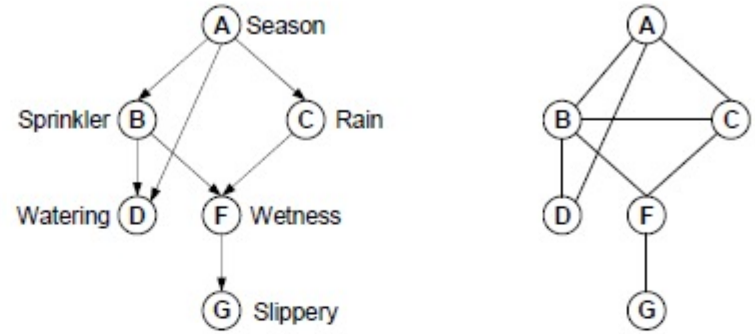
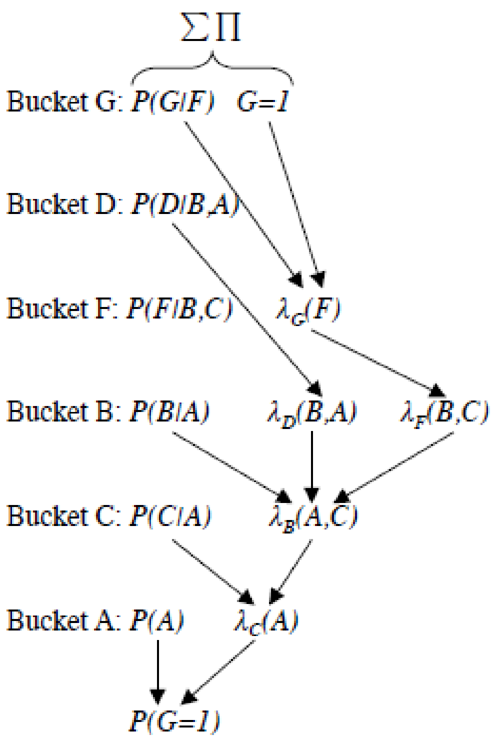


Figure 4.3: The bucket's output when processing along $d_2 = A, F, D, C, B, G$

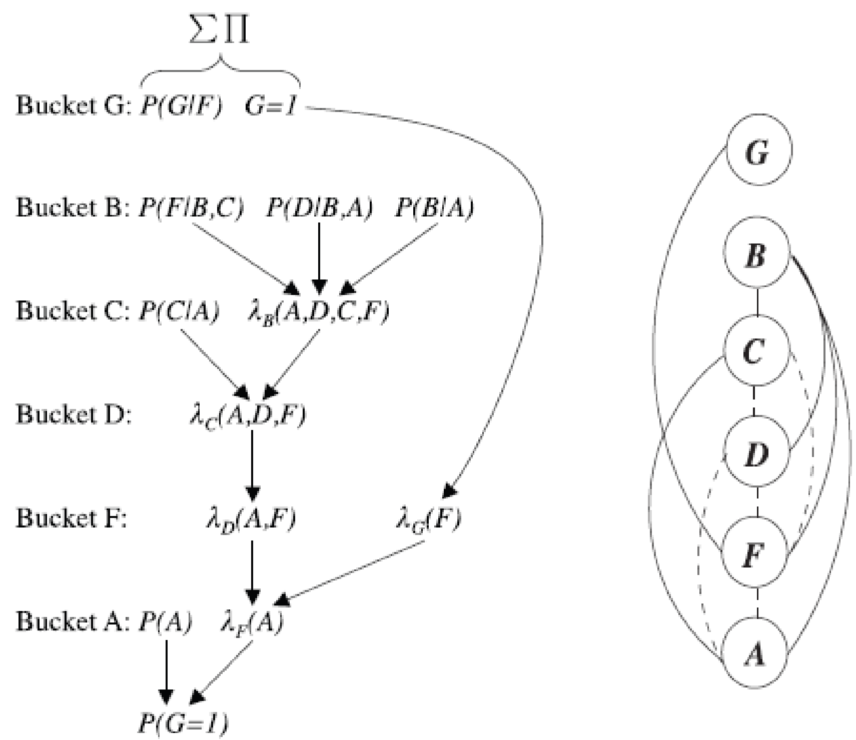
A Bayesian Network Processed Along 2 Orderings



(a) Directed acyclic graph (b) Moral graph



d1=A,C,B,F,D,G



(a) (b)

Figure 4.4: The bucket's output when processing along $d_2 = A, F, D, C, B, G$.

The Operation In a Bucket

- Multiplying functions
- Marginalizing (summing-out) functions

Combination of Cost Functions

A	B	f(A,B)
b	b	0.4
b	g	0.1
g	b	0
g	g	0.5

●

B	C	f(B,C)
b	b	0.2
b	g	0
g	b	0
g	g	0.8

A	B	C	f(A,B,C)
b	b	b	0.1
b	b	g	0
b	g	b	0
b	g	g	0.08
g	b	b	0
g	b	g	0
g	g	b	0
g	g	g	0.4

$= 0.1 \times 0.8$

Factors: Sum-Out Operation

The result of summing out variable X from factor $f(\mathbf{X})$

is another factor over variables $\mathbf{Y} = \mathbf{X} \setminus \{X\}$:

$$\left(\sum_X f\right)(\mathbf{y}) \stackrel{def}{=} \sum_x f(x, \mathbf{y})$$

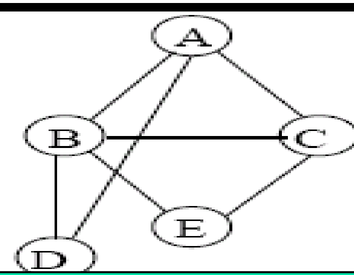
B	C	D	f_1
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

B	C	$\sum_D f_1$
true	true	1
true	false	1
false	true	1
false	false	1

	$\sum_B \sum_C \sum_D f_1$
$\top$	4

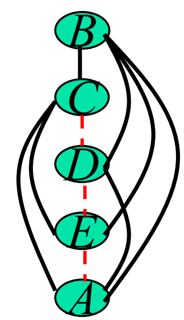
Thanks to Darwiche

Bucket Elimination and Induced Width



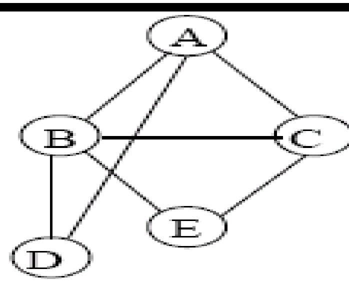
Ordering: a, e, d, c, b

$$
\begin{aligned}
bucket(B) &= \quad P(e|b,c), P(d|a,b), P(b|a) \\
bucket(C) &= \quad P(c|a) \;\; || \;\; \lambda_B(a,c,d,e) \\
bucket(D) &= \quad || \;\; \lambda_C(a,d,e) \\
bucket(E) &= \quad e = 0 \;\; || \;\; \lambda_D(a,c) \\
bucket(A) &= \quad P(a) \;\; || \;\; \lambda_E(a)
\end{aligned}
$$



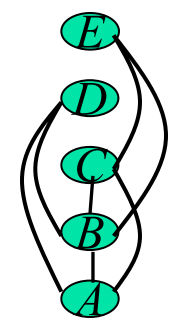
W=4*

Bucket Elimination and Induced Width



Ordering: a, b, c, d, e

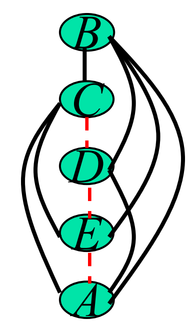
$$
\begin{aligned}
bucket(E) &= \quad P(e|b,c),\ \ e = 0 \\
bucket(D) &= \quad P(d|a,b) \\
bucket(C) &= \quad P(c|a) \quad || \ P(e=0|b,c) \\
bucket(B) &= \quad P(b|a) \quad || \quad \lambda_D(a,b), \lambda_C(b,c) \\
bucket(A) &= \quad P(a) \quad || \quad \lambda_B(a)
\end{aligned}
$$



$W^*=2$

Ordering: a, e, d, c, b

$$
\begin{aligned}
bucket(B) &= \quad P(e|b,c), P(d|a,b), P(b|a) \\
bucket(C) &= \quad P(c|a) \quad || \quad \lambda_B(a,c,d,e) \\
bucket(D) &= \quad\ || \ \lambda_C(a,d,e) \\
bucket(E) &= \quad e = 0 \quad || \ \lambda_D(a,c) \\
bucket(A) &= \quad P(a) \quad || \quad \lambda_E(a)
\end{aligned}
$$



$W^*=4$

ALGORITHM BE-BEL

Input: A belief network $\mathcal{B} = \langle \mathbf{X}, \mathbf{D}, \mathbf{P}_G, \prod \rangle$, an ordering $d = (X_1, \ldots, X_n)$; evidence e

output: The belief $P(X_1|e)$ and probability of evidence $P(e)$

1. Partition the input functions (CPTs) into $bucket_1$, …, $bucket_n$ as follows: **for** $i \leftarrow n$ **downto** 1, put in $bucket_i$ all unplaced functions mentioning X_i. Put each observed variable in its bucket. Denote by ψ_i the product of input functions in $bucket_i$.
2. **backward: for** $p \leftarrow n$ **downto** 1 **do**
3. **for** all the functions $\psi_{S_0}, \lambda_{S_1}, \ldots, \lambda_{S_j}$ in $bucket_p$ **do**

   If (observed variable) $X_p = x_p$ appears in $bucket_p$,

   assign $X_p = x_p$ to each function in $bucket_p$ and then

   put each resulting function in the bucket of the *closest* variable in its scope.

   else,
4. $\lambda_p \leftarrow \sum_{X_p} \psi_p \cdot \prod_{i=1}^{j} \lambda_{S_i}$
5. place λ_p in bucket of the latest variable in scope(λ_p),
6. **return** (as a result of processing $bucket_1$):

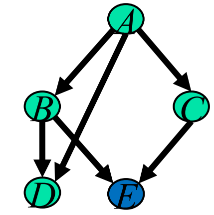
   $P(e) = \alpha = \sum_{X_1} \psi_1 \cdot \prod_{\lambda \in bucket_1} \lambda$

   $P(X_1|e) = \frac{1}{\alpha} \psi_1 \cdot \prod_{\lambda \in bucket_1} \lambda$

Figure 4.5: BE-bel: a sum-product bucket-elimination algorithm.

Belief Updating

Algorithm BE-bel [Dechter 1996]



$$p(A|E=0) = \alpha \sum_{e,d,c,b} p(A)\, p(b|A)\, p(c|A)\, p(d|A,b)\, p(e|b,c)\, \mathbb{1}[e=0]$$

$$\sum_b \prod \longleftarrow$$ *Elimination & combination operators*

bucket B: $\overbrace{p(b|A)\ p(d|b,A)\ p(e|b,c)}^{\sum_b \prod}$

bucket C: $p(c|A)$ $\lambda_{B\to C}(A, \underbrace{d, c, e})$

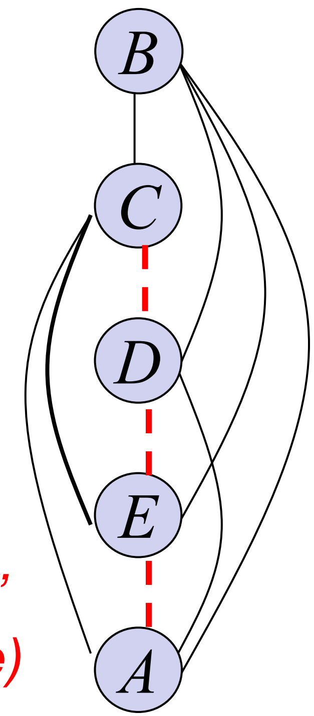
bucket D: $\lambda_{C\to D}(A,d,e)$

bucket E: $\mathbb{1}[E=0]\ \lambda_{D\to E}(A,e)$

bucket A: $p(A)$ $\lambda_{E\to A}(A)$

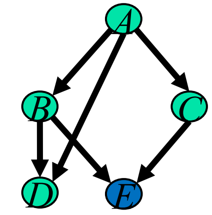
$p(E=0)$

W=4*
"induced width"
(max clique size)



$$p(A|E=0) = p(A,E=0)\, /\, p(E=0)$$

Bucket Elimination



Algorithm BE-bel *[Dechter 1996]*

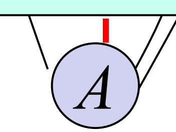
$$p(A|E=0) = \alpha \sum_{e,d,c,b} p(A)\, p(b|A)\, p(c|A)\, p(d|A,b)\, p(e|b,c)\, \mathbb{1}[e=0]$$

$\sum_b \prod$ ← *Elimination & combination operators*

Time and space exponential in the induced-width / treewidth

bucket A: $p(A)$ $\lambda_{E \to A}(A)$

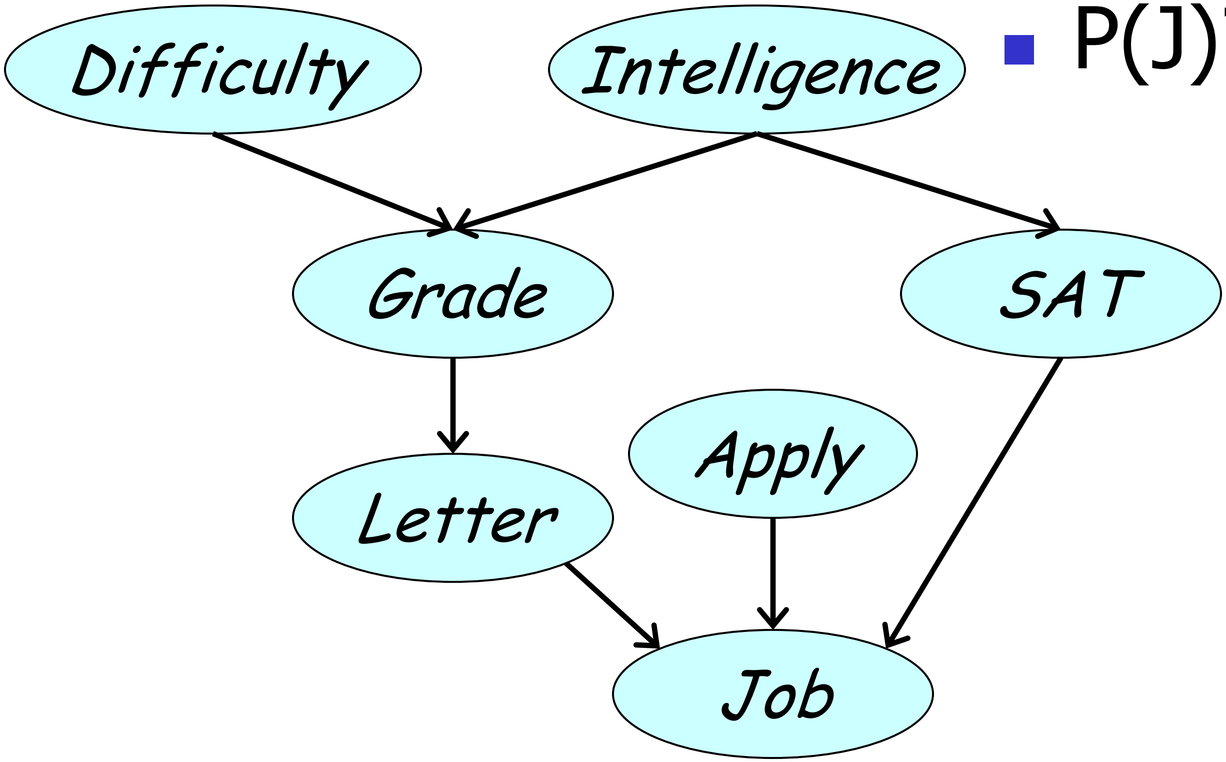
induced width (max clique size)



$p(E=0)$

$p(A|E=0) = p(A, E=0) \,/\, p(E=0)$

Student Network Example

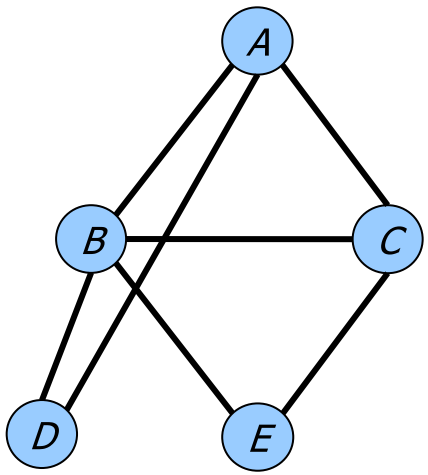


- P(J)?

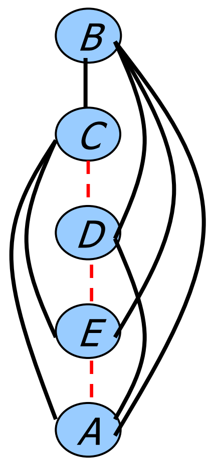
Induced Width (continued)

$w^*(d)$ – the induced width of the primal graph along ordering d

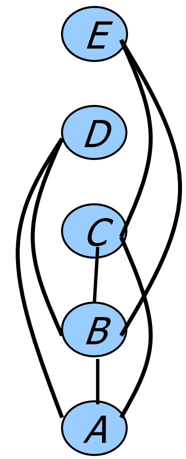
The effect of the ordering:



Primal (moraal) graph



$$w^*(d_1) = 4$$



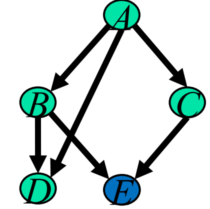
$$w^*(d_2) = 2$$

Inference for Probabilistic Networks

- Bucket elimination
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width

The Impact of Evidence?

Algorithm *BE-bel*



$$P(A \mid E=0) = \alpha \sum_{E=0,D,C,B} P(A)\cdot P(B\mid A)\cdot P(C\mid A)\cdot P(D\mid A,B)\cdot P(E\mid B,C)$$

$\sum_b \prod$ ← *Elimination operator*

bucket B: $\quad P(b|a) \quad P(d|b,a) \quad P(e|b,c)$ $\qquad$ *B=1*

bucket C: $\quad P(c|a) \quad \lambda^B(a,d,c,e)$

bucket D: $\quad \lambda^C(a,d,e)$

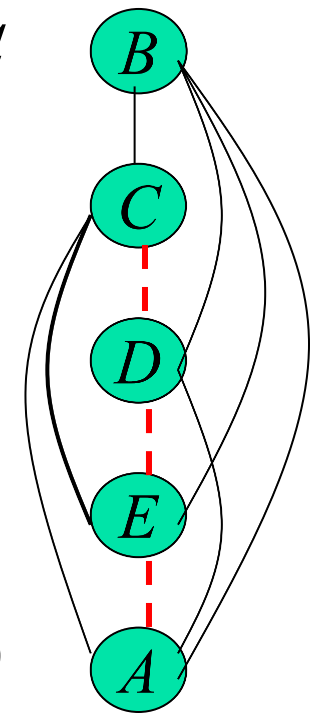
bucket E: $\quad e=0 \quad \lambda^D(a,e)$

bucket A: $\quad P(a) \quad \lambda^E(a)$

$P(e=0)$

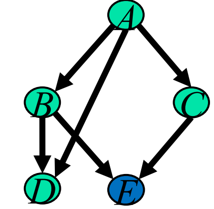
$P(a|e=0)$

W=4*
"induced width"
(max clique size)



The Impact of Evidence?

Algorithm *BE-bel*



$$P(A \mid E=0) = \alpha \sum_{E=0,D,C,B} P(A)\cdot P(B \mid A)\cdot P(C \mid A)\cdot P(D \mid A,B)\cdot P(E \mid B,C)$$

P(A|E=0,B=1)?

$\sum_b \prod$ ← Elimination operator

bucket B: $\underbrace{P(b|a) \quad P(d|b,a) \quad P(e|b,c)}$

B=1

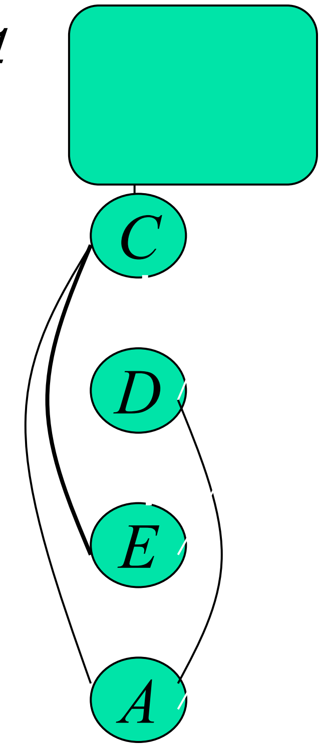
bucket C: $P(c|a)$ *P(e|b=1,c)*

bucket D: *P(d|b=1,a)*

bucket E: $e=0$

bucket A: $P(a)$ *P(b=1|a)*

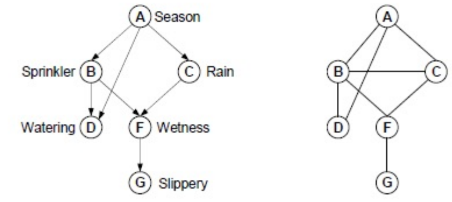
$\mathbf{P(e=0)}$



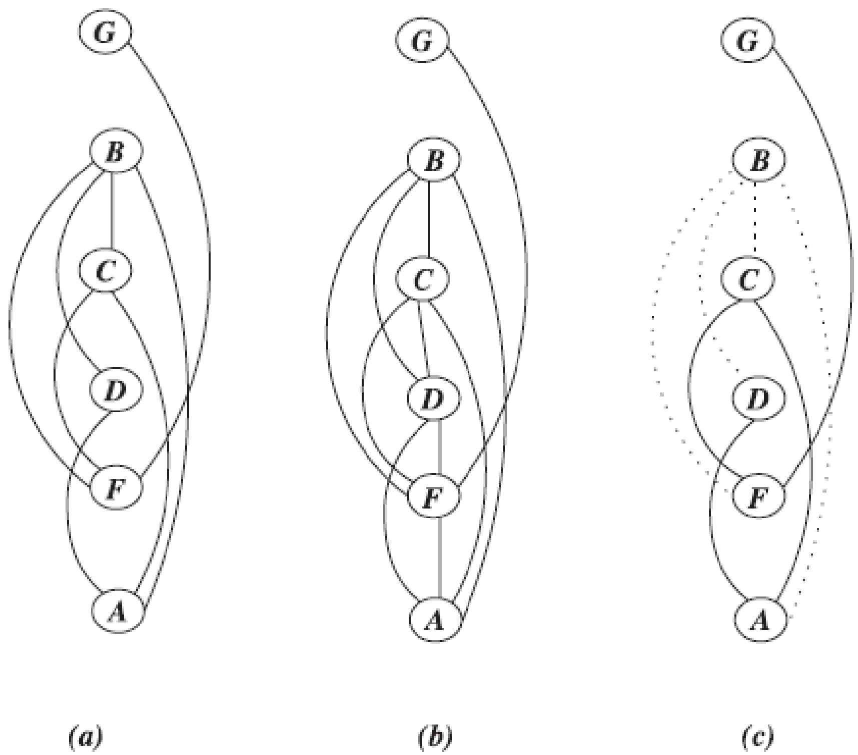
$\mathbf{P(a|e=0)}$

$$\mathbf{P(a|e=0)=\frac{P(a,e=0)}{P(e=0)}}$$

The Impact of Observations



(a) Directed acyclic graph (b) Moral graph



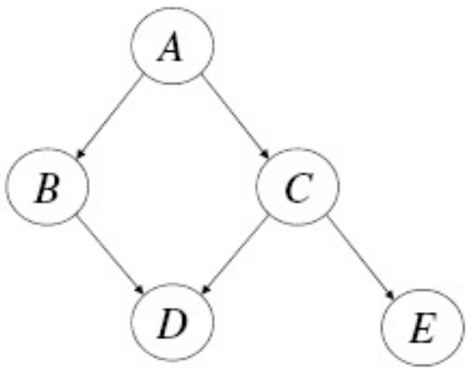
(a) (b) (c)

Figure 4.9: Adjusted induced graph relative to observing B.

Ordered graph *Induced graph* *Ordered conditioned graph*

Pruning Nodes: Example

Example of pruning irrelevant subnetworks



network structure

joint on B, E

joint on B

Pruning Nodes

Given a Bayesian network $\mathcal{N}$ and query $(\mathbf{Q}, \mathbf{e})$

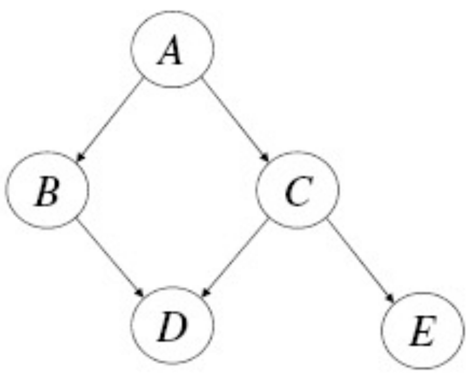
one can remove any leaf node (with its CPT) from the network as long as it does not belong to variables $\mathbf{Q} \cup \mathbf{E}$, yet not affect the ability of the network to answer the query correctly.

If $\mathcal{N}' = \text{pruneNodes}(\mathcal{N}, \mathbf{Q} \cup \mathbf{E})$

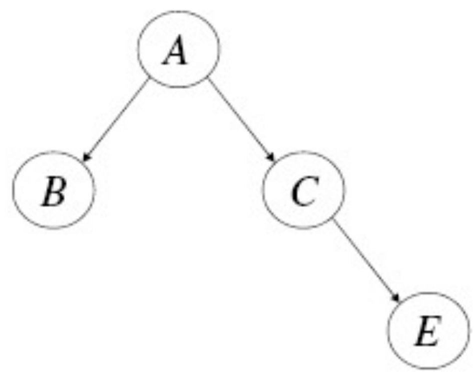
then $\Pr(\mathbf{Q}, \mathbf{e}) = \Pr'(\mathbf{Q}, \mathbf{e})$, where $\Pr$ and $\Pr'$ are the probability distributions induced by networks $\mathcal{N}$ and $\mathcal{N}'$, respectively.

Pruning Nodes: Example

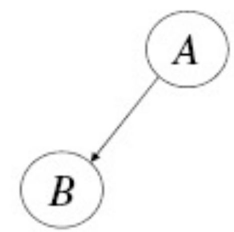
Example of pruning irrelevant subnetworks



network structure



joint on B, E

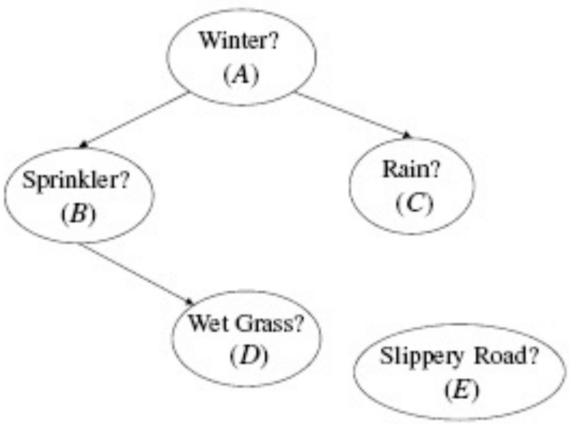


joint on B

Pruning Edges: Example

Example of pruning edges due to evidence or conditioning

A	B	$\Theta_{B\|A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25



A	C	$\Theta_{C\|A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

A	Θ_A
true	.6
false	.4

B	D	$\sum_C \Theta_{D\|BC}^{C=\text{false}}$
true	true	.9
true	false	.1
false	true	0
false	false	1

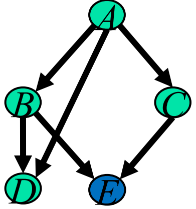
E	$\sum_C \Theta_{E\|C}^{C=\text{false}}$
true	0
false	1

Evidence **e** : $C = \text{false}$

Inference for Probabilistic Networks

- Bucket elimination
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width

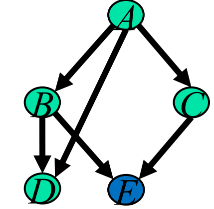
$\text{MPE} = \max_{\bar{x}} P(\bar{x})$



$\sum$ is replaced by ***max*** :

$$MPE = \max_{a,e,d,c,b} P(a)P(c \mid a)P(b \mid a)P(d \mid a,b)P(e \mid b,c)$$

$\text{MPE} = \max_{\bar{x}} P(\bar{x})$



$\sum$ is replaced by ***max*** :

$$MPE = \max_{a,e,d,c,b} P(a)P(c\,|\,a)P(b\,|\,a)P(d\,|\,a,b)P(e\,|\,b,c)$$

$\max_b \prod$ ← *Elimination operator*

bucket B: $\underbrace{P(b|a) \quad P(d|b,a) \quad P(e|b,c)}$

bucket C: $P(c|a) \quad h^B(\underbrace{a,d,c,e})$

bucket D: $h^C(a,d,e)$

bucket E: $e=0 \quad h^D(a,e)$

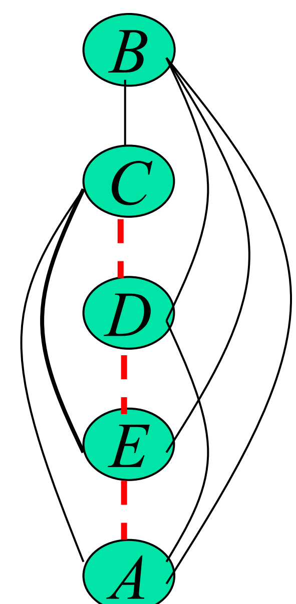
bucket A: $P(a) \quad h^E(a)$

MPE

W=4*

"induced width"

(max clique size)



Generating the MPE-tuple

5. $b' = \arg\max_b P(b \mid a') \times P(d' \mid b, a') \times P(e' \mid b, c')$

4. $c' = \arg\max_c P(c \mid a') \times h^B(a', d', c, e')$

3. $d' = \arg\max_d h^C(a', d, e')$

2. $e' = 0$

1. $a' = \arg\max_a P(a) \cdot h^E(a)$

B: $P(b|a)$ $P(d|b,a)$ $P(e|b,c)$

C: $P(c|a)$ $h^B(a,d,c,e)$

D: $h^C(a,d,e)$

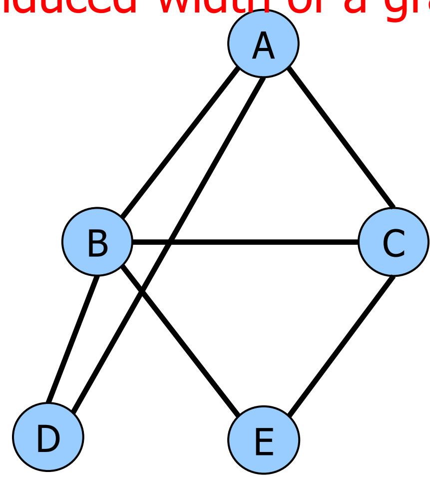
E: $e=0$ $h^D(a,e)$

A: $P(a)$ $h^E(a)$

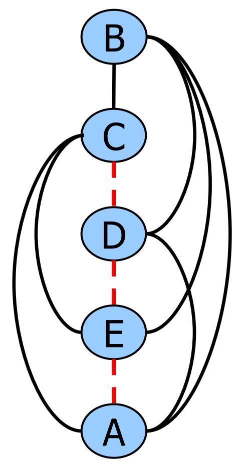
Return (a', b', c', d', e')

Induced Width

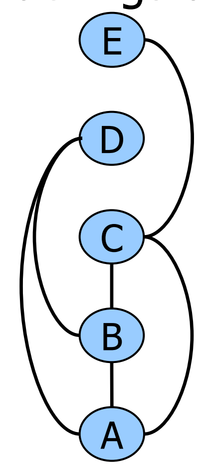
- Width is the max number of parents in the ordered graph
- Induced-width is the width of the induced ordered graph: recursively connecting parents going from last node to first.
- Induced-width w*(d) is the max induced-width over all nodes in ordering d
- Induced-width of a graph, w* is the min w*(d) over all orderings d



primal graph



$w^*(d_1) = 4$



$w^*(d_2) = 2$

Complexity of Bucket Elimination

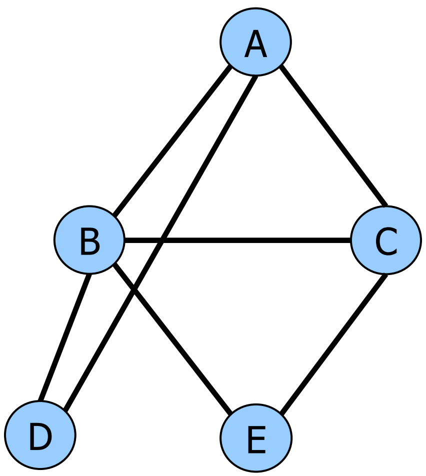
Bucket-Elimination is ***time*** *and* ***space***

$$O\left(r \exp(w_d^*)\right)$$

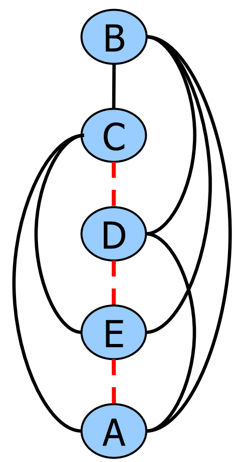
w_d^* *: the induced width of the primal graph along ordering d*

r = number of functions

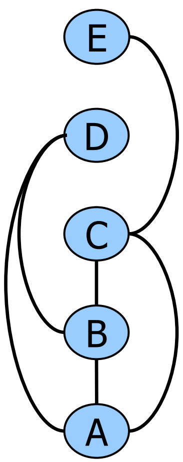
The effect of the ordering:



primal graph



$w^*(d_1) = 4$

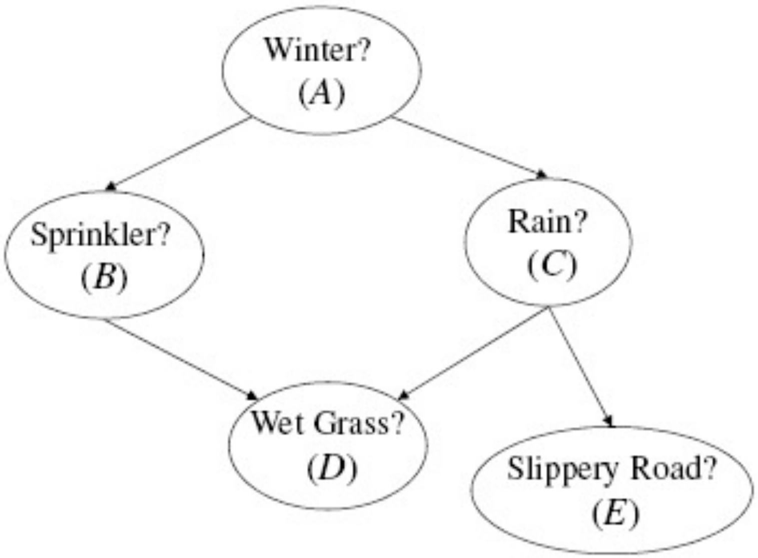


$w^*(d_2) = 2$

Finding smallest induced-width is hard!

A Bayesian Network

Example with mpe?



A	Θ_A
true	.6
false	.4

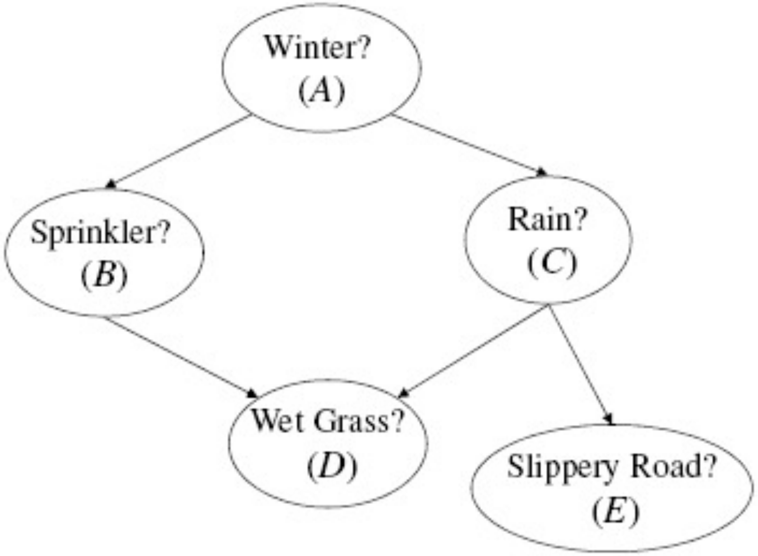
A	B	$\Theta_{B\|A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25

A	C	$\Theta_{C\|A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

B	C	D	$\Theta_{D\|BC}$
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

C	E	$\Theta_{E\|C}$
true	true	.7
true	false	.3
false	true	0
false	false	1

Try to compute MPE when E=0



A	Θ_A
true	.6
false	.4

A	B	$\Theta_{B\mid A}$
true	true	.2
true	false	.8
false	true	.75
false	false	.25

A	C	$\Theta_{C\mid A}$
true	true	.8
true	false	.2
false	true	.1
false	false	.9

B	C	D	$\Theta_{D\mid BC}$
true	true	true	.95
true	true	false	.05
true	false	true	.9
true	false	false	.1
false	true	true	.8
false	true	false	.2
false	false	true	0
false	false	false	1

C	E	$\Theta_{E\mid C}$
true	true	.7
true	false	.3
false	true	0
false	false	1

Cost Networks

$$P(a,b,c,d,f,g) = P(a)P(b|a)P(c|a)P(f|b,c)P(d|a,b)P(g|f)$$

becomes

$$C(a,b,c,d,e) = -logP = C(a) + C(b,a) + C(c,a) + C(f,b,c) + C(d,a,b) + C(g,f)$$

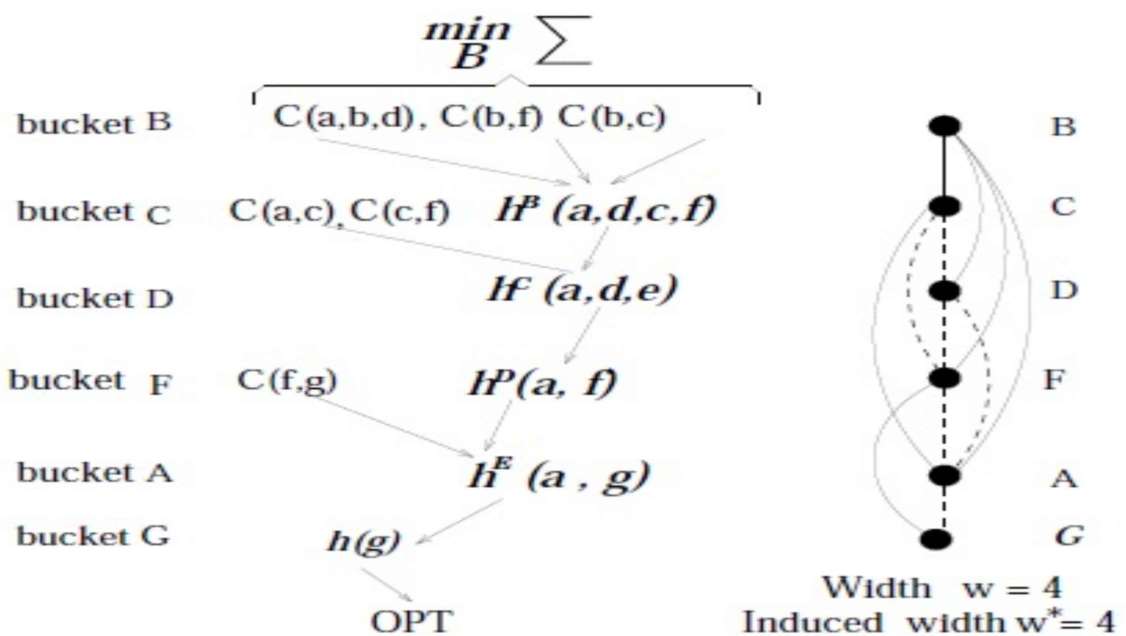


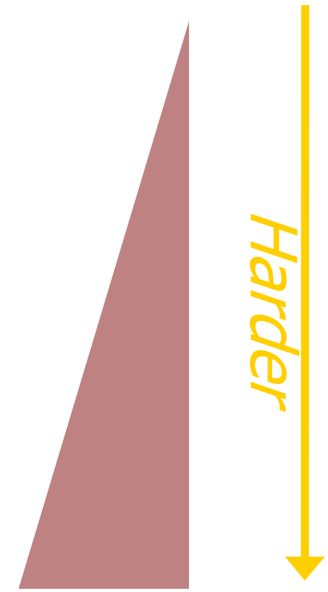
Figure 5.12: Schematic execution of BE-Opt

Inference for probabilistic networks

- Bucket elimination (Dechter chapter 4)
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
  - Influence diagrams ?
- Induced-Width (Dechter, Chapter 3.4)

Marginal Map

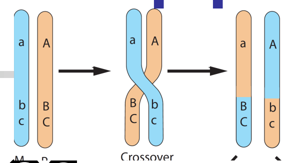
Max-Inference	$f(\mathbf{x}^*) = \max_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Sum-Inference	$Z = \sum_{\mathbf{x}} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$
Mixed-Inference	$f(\mathbf{x}_M^*) = \max_{\mathbf{x}_\mathrm{M}} \sum_{\mathbf{x}_S} \prod_{\alpha} f_\alpha(\mathbf{x}_\alpha)$



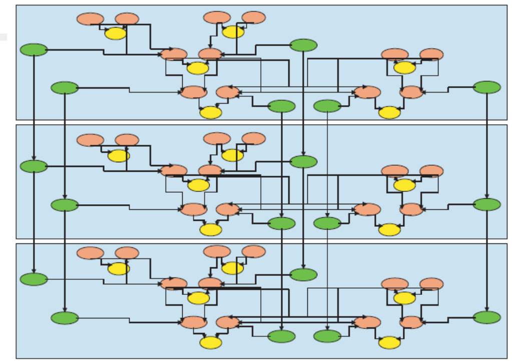
- **NP-hard**: exponentially many terms

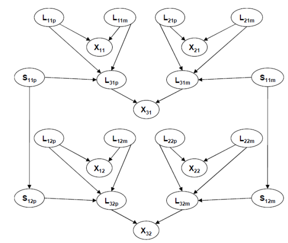
Example for MMAP Applications

- Haplotype in Family pedigrees



6 people, 3 markers





- Coding networks

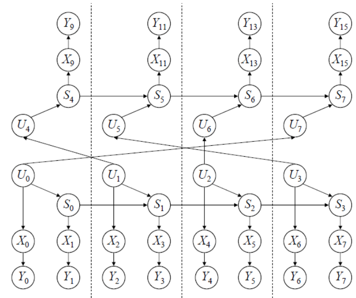
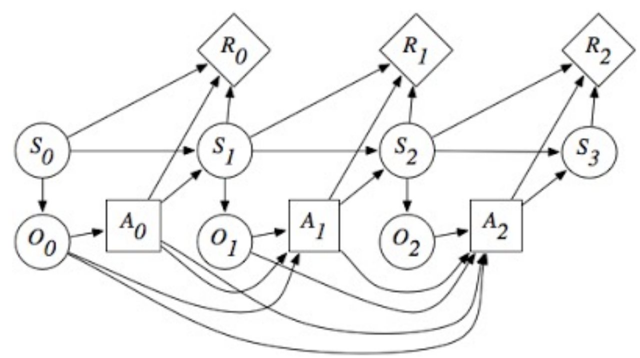
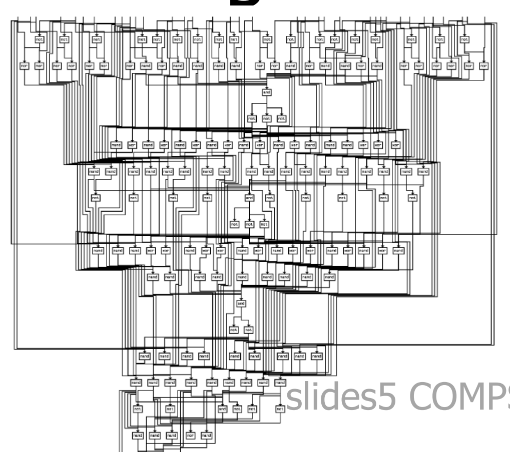


Figure 5.24: A Bayesian network for a turbo code.

- Probabilistic planning

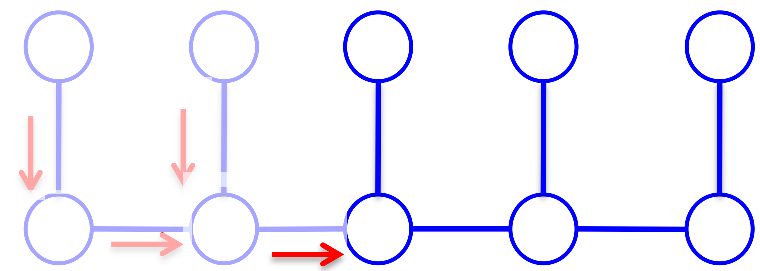


- Diagnosis



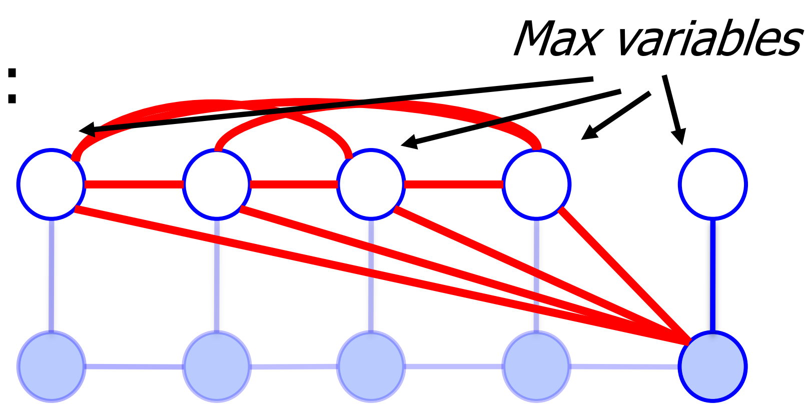
Marginal MAP is Not Easy on Trees

- Pure MAP or summation tasks
  - Dynamic programming
  - Ex: efficient on trees



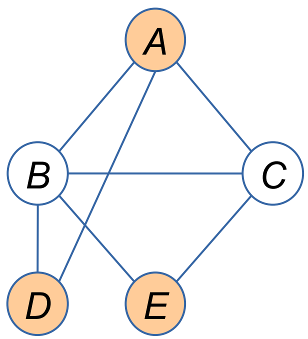
- Marginal MAP
  - Operations do not commute:
  - Sum must be done first!

$$\sum \max \neq \max \sum$$



Bucket Elimination for MMAP

Bucket Elimination



$\mathbf{X}_M = \{A, D, E\}$

$\mathbf{X}_S = \{B, C\}$

$$\max_{\mathbf{X}_M} \sum_{\mathbf{X}_S} P(\mathbf{X})$$

constrained elimination order

SUM

B: $\underbrace{f(A,B)\, f(B,C)\, f(B,D) f(B,E)}$

Σ_B

C: $\underbrace{\lambda^B(A,C,D,E) f(A,C)\ f(C,E)}$

Σ_C

MAX

D: $\underbrace{\lambda^C(A,D,E)\ f(A,D)}$

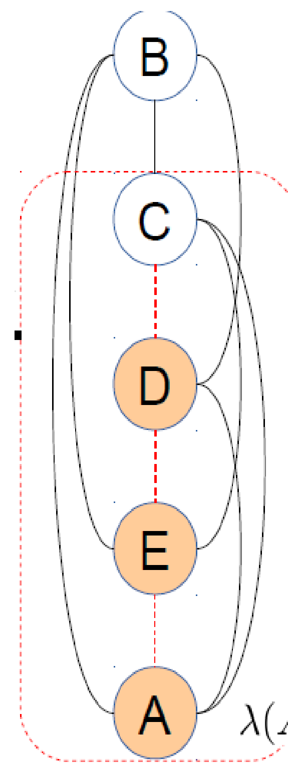
max_D

E: $\underbrace{\lambda^D(A,E)}$

max_E

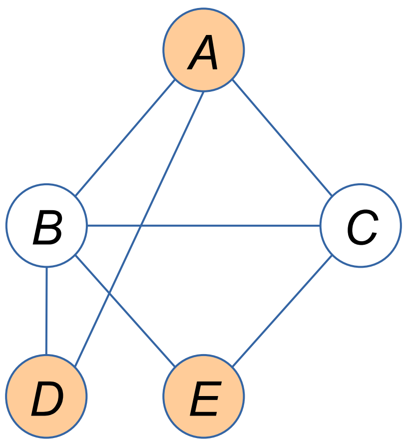
A: $\underbrace{\lambda^E(A)}$

MAP is the marginal MAP value*



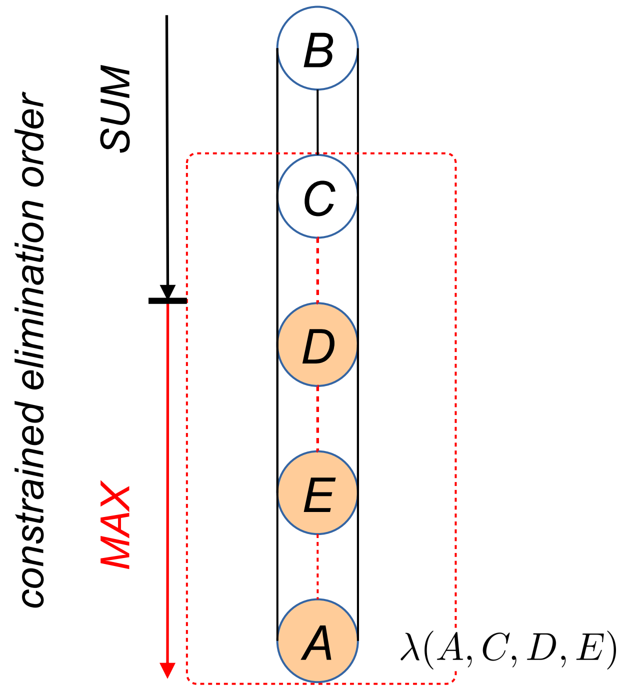
$\lambda(A,C,D,E)$

Why is MMAP harder?



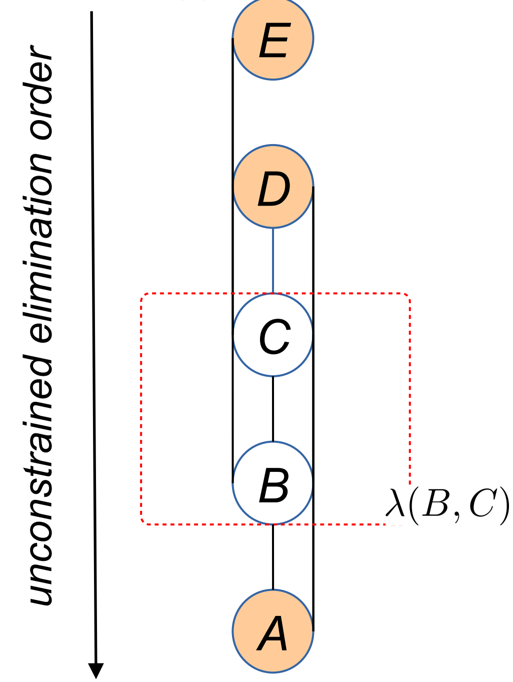
$$\mathbf{X}_M = \{A, D, E\}$$
$$\mathbf{X}_S = \{B, C\}$$

exact



$$w^* = 4$$

upper bound



$$w^* = 2$$

In practice, constrained induced is much larger!

$$\max_X \sum_Y \phi \leq \sum_Y \max_X \phi$$

(Park & Darwiche, 2003)
(Yuan & Hansen, 2009)

Complexity of Bucket-Elimination

- **Theorem:**

BE is $O(n \exp(w^*+1))$ time and $O(n \exp(w^*))$ space, when w^* is the induced-width of the moral graph along d when evidence nodes are processed (edges from evidence nodes to earlier variables are removed.)

More accurately: $O(r \exp(w^*(d))$ where r is the number of CPTs. For Bayesian networks $r=n$. For Markov networks?

Inference with Markov Networks

- Undirected graphs with potentials on cliques
- Query: find *partition function. Same as* probability of the evidence in a Bayesian network.
- The joint probability distribution of a Markov network is defined by:

$$P(x) = \frac{1}{Z} \sum_{x \in \mathcal{D}} \Pi_{C \in \mathcal{C}} \Psi_C(x_C)$$

BE is equally applicable

$$Z = \Sigma_x \Pi_{C \in \mathcal{C}} \Psi_C(x_C) \tag{2.2}$$

For example. A markov network over the moral graph in Figure 2.4(b) is defined by:

$$P(a,b,c,d,f,g) = \frac{\Psi(a,b,c) \cdot \Psi(b,c,f) \cdot \Psi(a,b,d) \cdot \Psi(f,g)}{Z} \tag{2.3}$$

where,

$$Z = \sum_{a,b,c,d,e,f,g} \Psi(a,b,c) \cdot \Psi(b,c,f) \cdot \Psi(a,b,d) \cdot \Psi(f,g) \tag{2.4}$$

Inference for probabilistic networks

- Bucket elimination
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width (Dechter 3.4,3.5)

Finding a Small Induced-Width

- NP-complete
- A tree has induced-width of ?
- Greedy algorithms:
  - Min width
  - Min induced-width
  - Max-cardinality and chordal graphs
  - Fill-in (thought as the best)
- Anytime algorithms
  - Search-based [Gogate & Dechter 2003]
  - Stochastic (CVO) [Kask, Gelfand & Dechter 2010]

Finding a Small Induced-Width

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Min-width Ordering

MIN-WIDTH (MW)

input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$

output: A min-width ordering of the nodes $d = (v_1, ..., v_n)$.

1. **for** $j = n$ to 1 by -1 do
2. $r \leftarrow$ a node in G with smallest degree.
3. put r in position j and $G \leftarrow G - r$.
   (Delete from V node r and from E all its adjacent edges)
4. **endfor**

Proposition: *algorithm min-width finds a min-width ordering of a graph*

What is the Complexity of MW?

O(e)

Greedy Orderings Heuristics

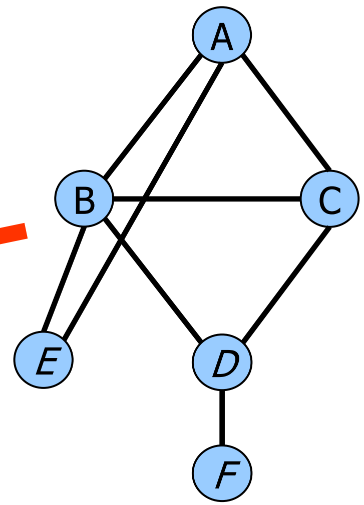
- Min-induced-width
  - From last to first, pick a node with smallest width, then connect parent and remove

- Min-Fill
  - From last to first, pick a node with smallest fill-edges

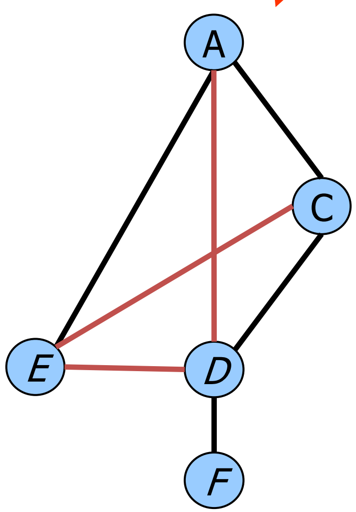
Complexity? $O(n^3)$

Min-Fill Heuristic

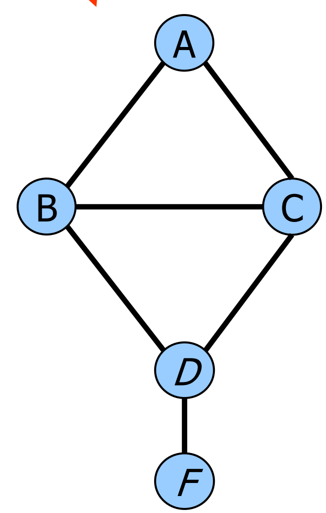
- Select the variable that creates the fewest "fill-in" edges



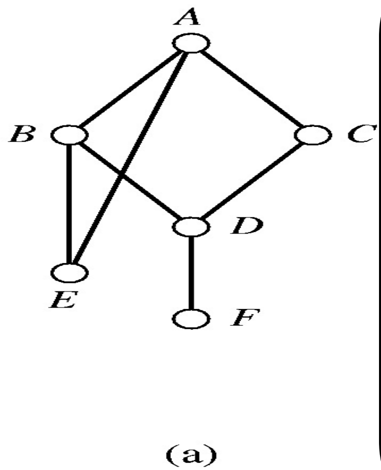
Eliminate B next?
Connect neighbors
"Fill-in" = 3:
(A,D), (C,E), (D,E)



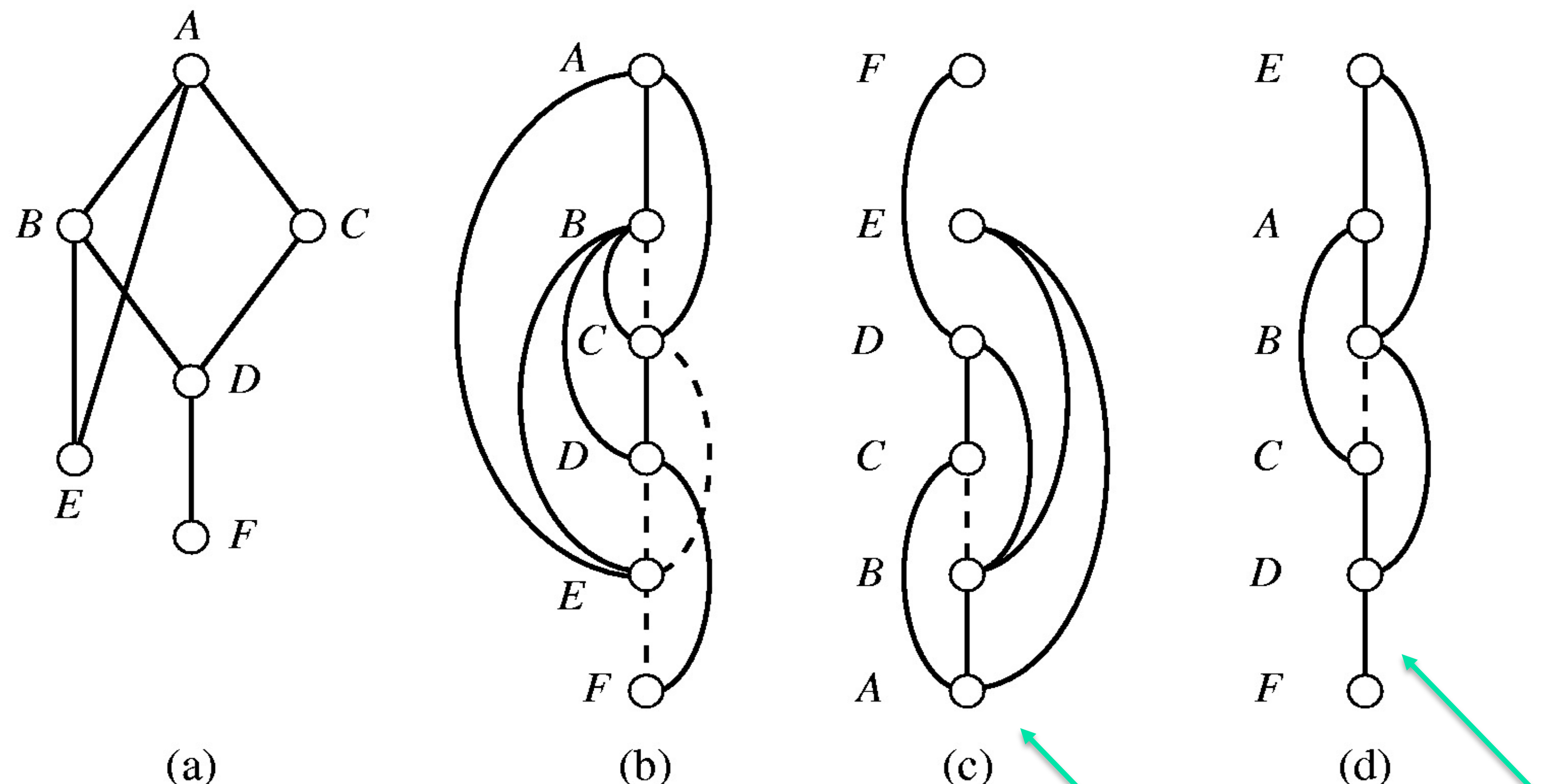
Eliminate E next?
Neighbors already connected
"Fill-in" = 0



Example



Different Induced-Graphs

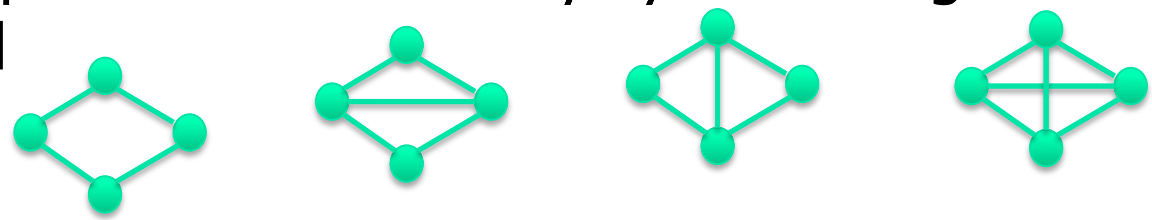


A Min-fill ordering

A Miw ordering

Chordal Graphs

- A graph is chordal if every cycle of length at least 4 has a chord



- Deciding chordality by max-cardinality ordering:
  - from 1 to n, always assigning a next node connected to a largest set of previously selected nodes.
- A graph along max-cardinality order has no fill-in edges iff it is chordal.
- The maximal cliques of chordal graphs form a tree

[Tarjan & Yanakakis 1980]

Greedy Orderings Heuristics

- Min-Induced-width
  - From last to first, pick a node with smallest width

- Min-Fill
  - From last to first, pick a node with smallest fill-edges

    Complexity? $O(n^3)$

- Max-Cardinality search [Tarjan & Yanakakis 1980]
  - From **first to last**, pick a node with largest neighbors already ordered. *Complexity?* $O(n+m)$

Max-cardinality ordering

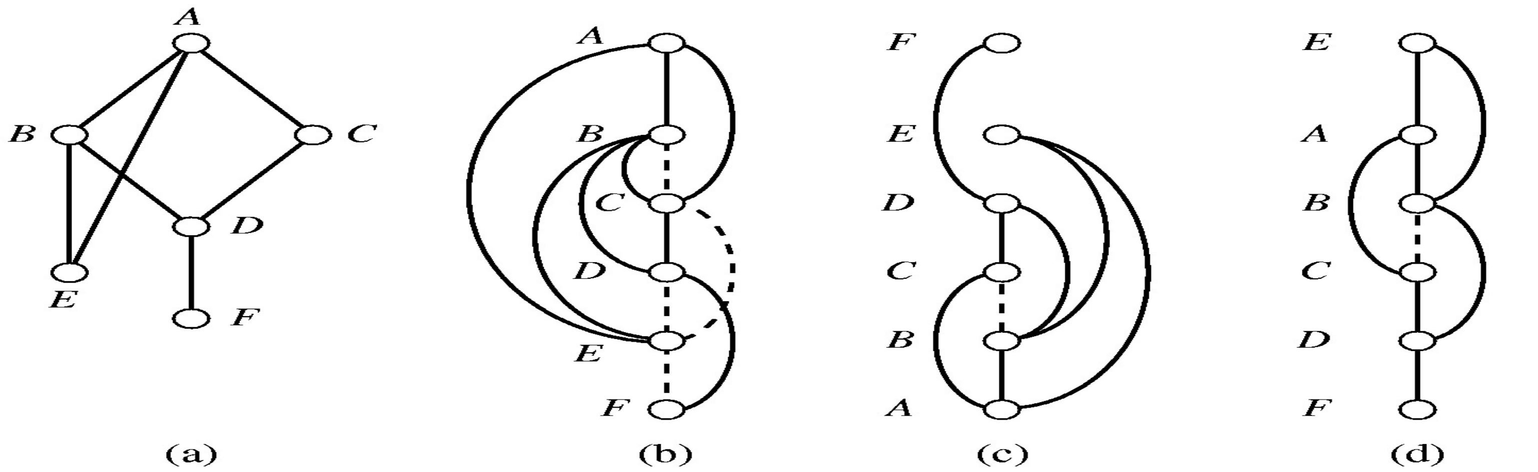
MAX-CARDINALITY (MC)

input: a graph $G = (V, E)$, $V = \{v_1, ..., v_n\}$
output: An ordering of the nodes $d = (v_1, ..., v_n)$.
1. Place an arbitrary node in position 0.
2. **for** $j = 1$ to n do
3. $r \leftarrow$ a node in G that is connected to a largest subset of nodes in positions 1 to $j - 1$, breaking ties arbitrarily.
4. **endfor**

Proposition 5.3.3 *[56] Given a graph $G = (V, E)$ the complexity of max-cardinality search is $O(n + m)$ when $|V| = n$ and $|E| = m$.*

Example

We see again that *G* in the Figure (a) is not chordal since the parents of *A* are not connected in the max-cardinality ordering in Figure (d). If we connect *B* and *C*, the resulting induced graph is chordal.



Which Greedy Algorithm is Best?

- Min-Fill, prefers a node who add the least number of fill-in arcs.
- Empirically, fill-in is the best among the greedy algorithms (MW,MIW,MF,MC)
- Complexity of greedy orderings?
- MW is O(e), MIW: O(n^3) MF O(n^3) MC is O(e+n)

K-trees

Definition 5.3.4 (k-trees) *A subclass of chordal graphs are k-trees. A k-tree is a chordal graph whose maximal cliques are of size $k+1$, and it can be defined recursively as follows: (1) A complete graph with k vertices is a k-tree. (2) A k-tree with r vertices can be extended to $r+1$ vertices by connecting the new vertex to all the vertices in any clique of size k. A partial k-tree is a k-tree having some of its arcs removed. Namely it will clique of size smaller than k.*

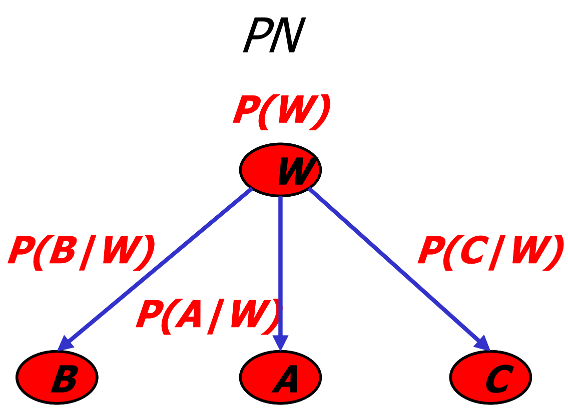
Finding a Small Induced-Width

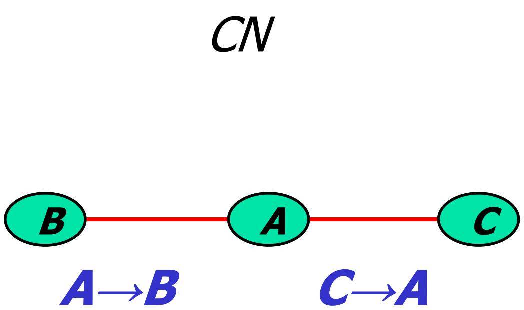
- NP-complete
- A tree has induced-width of ?
- Greedy algorithms:
  - Min width (MW)
  - Min induced-width (MIW)
  - Max-cardinality and chordal graphs (MC)
  - Min-Fill (thought as the best) (MIN-FILL)
- Anytime algorithms
  - Search-based [Gogate & Dechter 2003]
  - Stochastic (CVO) [Kask, Gelfand & Dechter 2010]

Inference for probabilistic networks

- Bucket elimination (Dechter chapter 4)
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width (Dechter, Chapter 3.4)
- Mixed networks
- Influence diagrams ?

Party Example





Query:
Is it likely that Chris goes to the party if Becky does not but the weather is bad?

Semantics?

Algorithms?

$$P(C, \neg B \mid w = bad, A \rightarrow B, C \rightarrow A)$$

Bucket Elimination for Mixed Networks

The CPE (constraint probability evaluation) query

$$P_{\mathcal{B}}(\varphi) = \sum_{\mathbf{x}_\varphi \in Mod(\varphi)} P(\mathbf{x}_\varphi)$$

Using the belief network product form we get:

$$P_{\mathcal{B}}(\varphi) = \sum_{\{\mathbf{x} | \mathbf{x}_\varphi \in Mod(\varphi)\}} \prod_{i=1}^{n} P(x_i | \mathbf{x}_{pa_i}).$$

P((C → B) and P(A → C))

Bucket-Elimination example for a Mixed Network

$\varphi = (B \vee C), (G \vee D), (\sim D \vee \sim B)$

Bucket G: $P(G|F,D)$ $\quad (G \vee D)$

Bucket F: $P(F|B,C)$ $\quad \lambda_G(F,D)$

Bucket D: $P(D|A,B)$ $\quad (\neg D \vee \neg B)$ $\quad \lambda_F(B,C,D)$

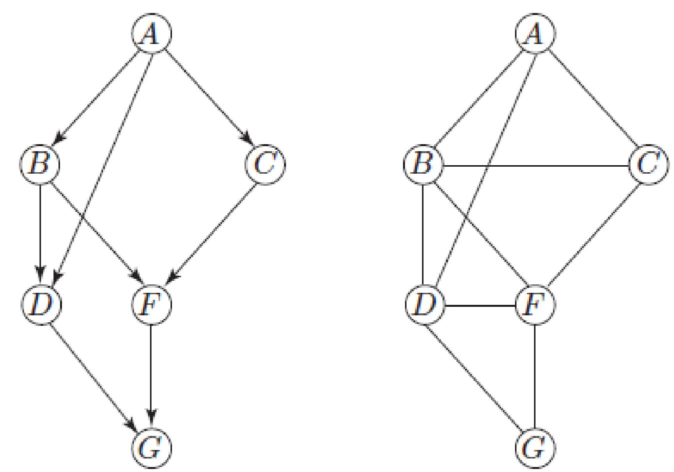
Bucket B: $P(B|A)$ $\quad (B \vee C)$ $\quad \lambda_D(A,B,C)$

Bucket C: $P(C|A)$ $\quad \lambda_B(A,C)$

Bucket A: $P(A)$ $\quad \lambda_C(A)$

$P(\varphi)$

Figure 4.18: Execution of BE-CPE.



(a) Directed Acyclic Graph (b) Moral Graph

Bucket-Elimination example for a Mixed Network

$\varphi = (B \vee C), (G \vee D), (\sim D \vee \sim B)$

In $Bucket G$: $\lambda_G(f,d) = \sum_{\{g|g \vee d = true\}} P(g|f)$

In $Bucket_F$: $\lambda_F(b,c,d) = \sum_f P(f|b,c)\lambda_G(f,d)$

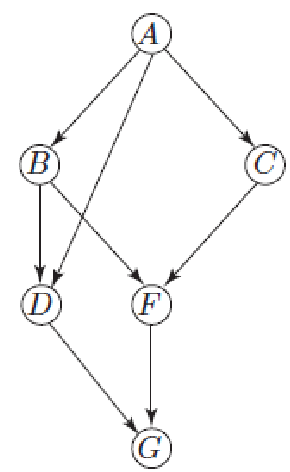
In $Bucket_D$: $\lambda_D(a,b,c) = \sum_{\{d|\neg d \vee \neg b = true\}} P(d|a,b)\lambda_F(b,c,d)$

In $Bucket_B$: $\lambda_B(a,c) = \sum_{\{b|b \vee c = true\}} P(b|a)\lambda_D(a,b,c)\lambda_F(b,c)$

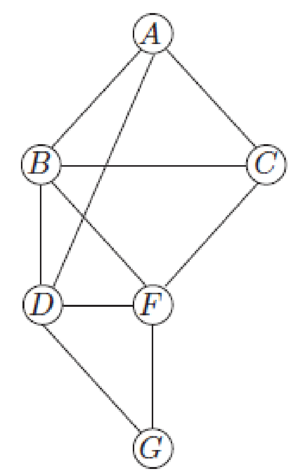
In $Bucket_C$: $\lambda_C(a) = \sum_c P(c|a)\lambda_B(a,c)$

In $Bucket_A$: $\lambda_A = \sum_a P(a)\lambda_C(a)$

$P(\varphi) = \lambda_A$.



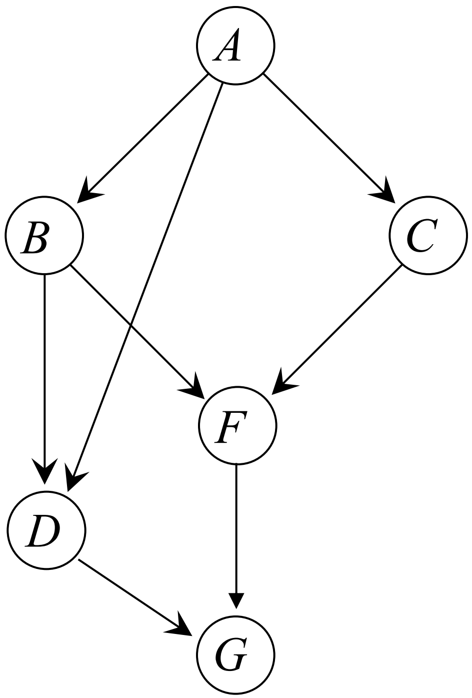
(a) Directed Acyclic Graph

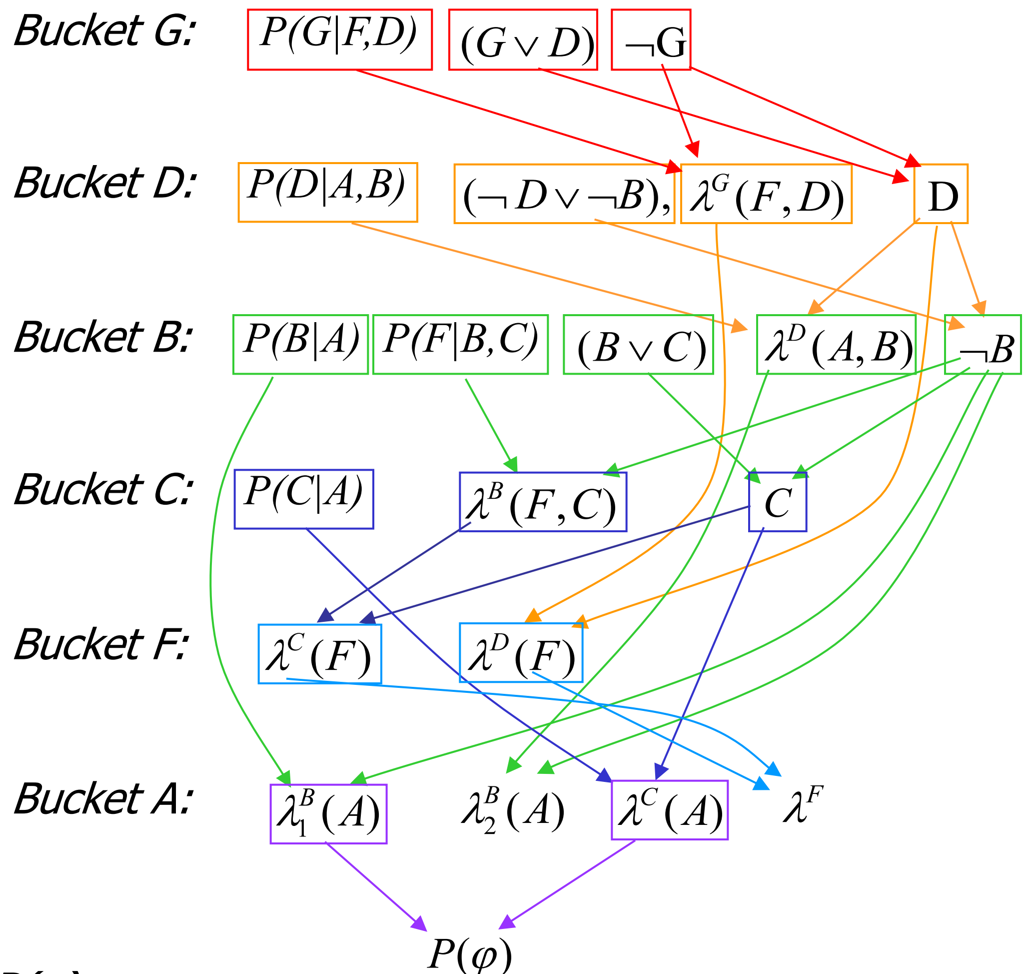


(b) Moral Graph

Trace of Elim-CPE

Adding evidence G=0

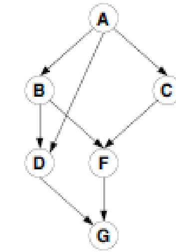




Belief network P(g,f,d,c,b,a)
=P(g|f,d)P(f|c,b)P(d|b,a)P(b|a)P(c|a)P(a)

Bucket-Elimination example for a Mixed Network

$\varphi = (B \vee C), (G \vee D), (\sim D \vee \sim B)$



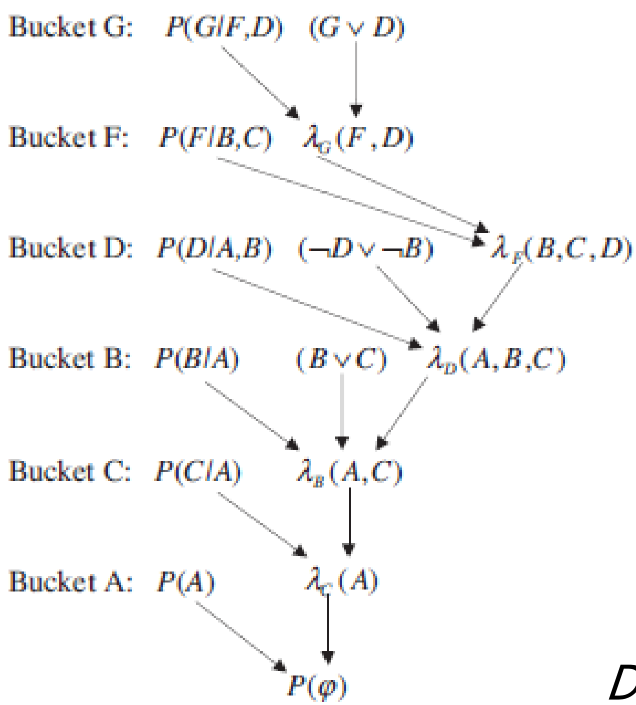


Figure 4.15: Execution of BE-CPE.

Different orderings

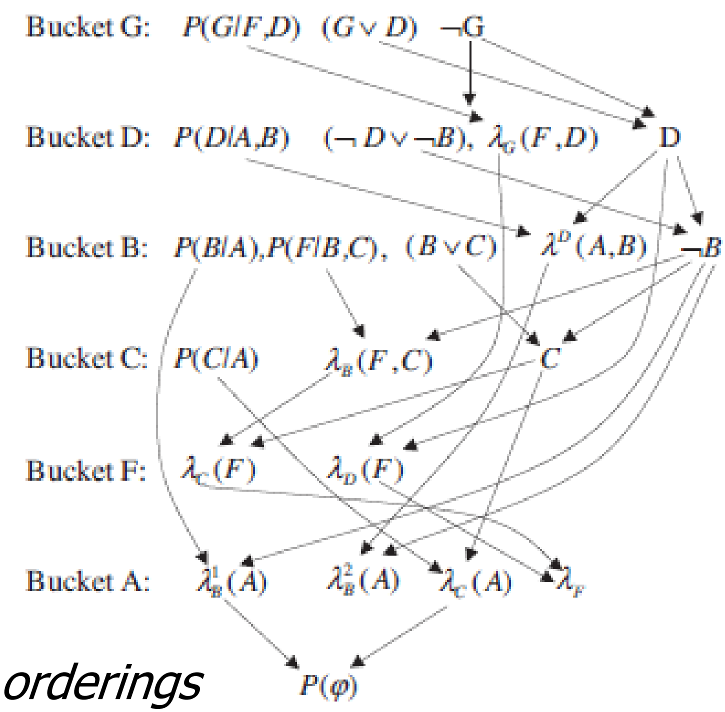


Figure 4.16: Execution of BE-CPE (evidence $\neg G$).

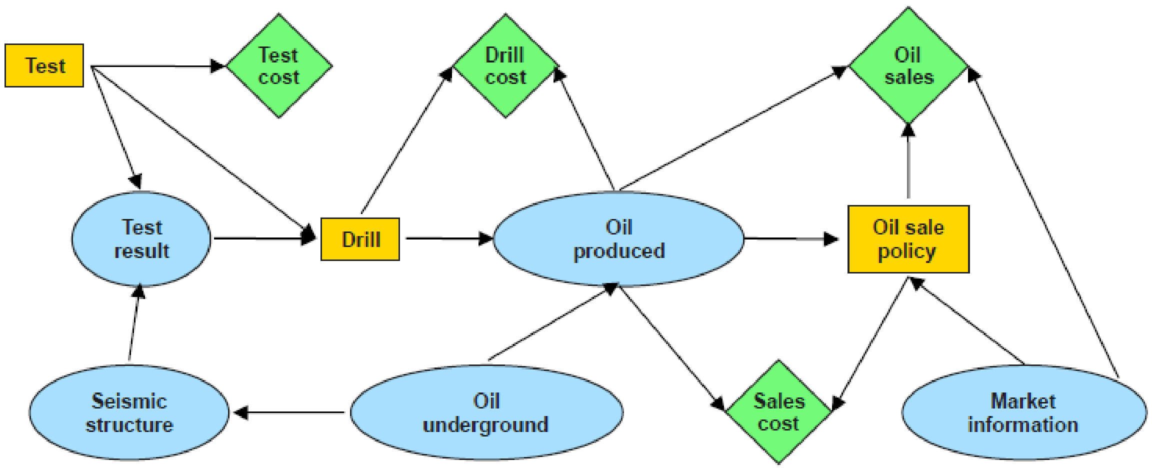
Inference for probabilistic networks

- Bucket elimination (Dechter chapter 4)
  - Belief-updating, P(e), partition function
  - Marginals, probability of evidence
  - The impact of evidence
  - for MPE (→MAP)
  - for MAP (→ Marginal Map)
- Induced-Width (Dechter, Chapter 3.4)
- Mixed networks
- Influence diagrams ?

Ex: "oil wildcatter"

e.g., [Raiffa 1968; Shachter 1986]

- Influence diagram:



- Three actions: test, drill, sales policy
- Chance variables:

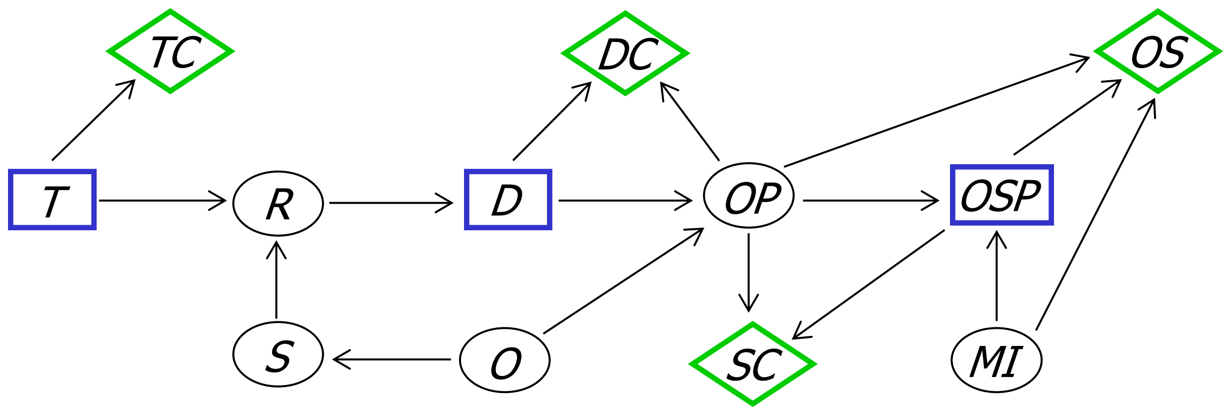
  P(oil) P(seismic|oil) P(result | seismic, test) P(produced | oil, drill) P(market)

- Utilities capture costs of actions, rewards of sale

  Oil sales - Test cost - Drill cost - Sales cost

Influence Diagrams

Influence diagram *ID = (X,D,P,R).*



Chance variables $X = X_1,\ldots,X_n$ ***over domains.***

Decision variables $D = D_1,\ldots,D_m$

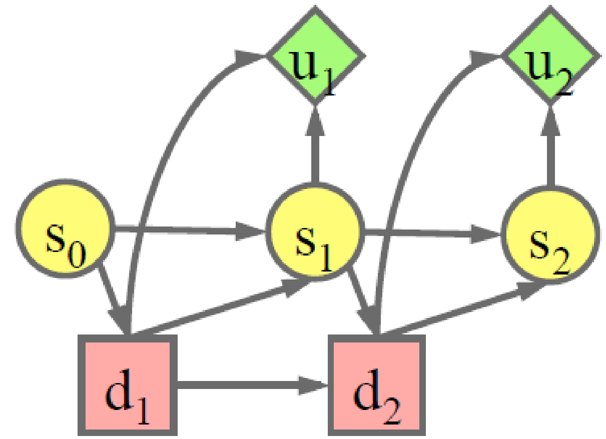
CPT's for chance variables $P_i = P(X_i \mid pa_i), i = 1..n$

Reward components $R = \{r_1,\ldots,r_j\}$

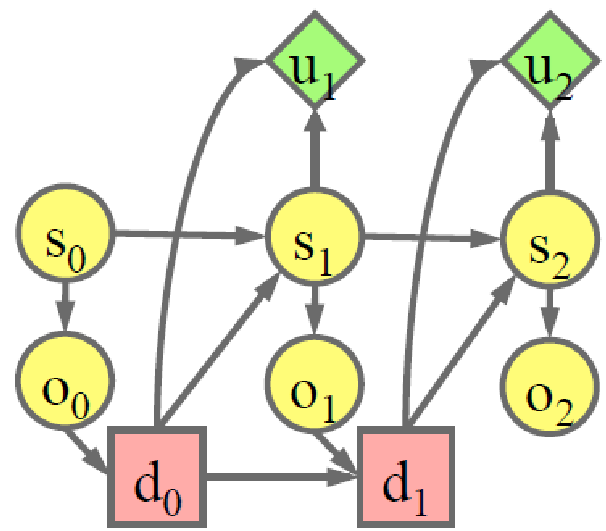
Utility function $u = \sum_i r_i$

Common examples

- Markov decision process
  - Markov chain state sequence
  - Actions "di" influence state transition
  - Rewards based on action, new state
  - Temporally homogeneous



- Partially observable MDP
  - Hidden Markov chain state sequence
  - Generate observations
  - Actions based on observations



Influence Diagrams (continue)

A decision rule *for* D_i *is a mapping:* $\delta_i : \Omega_{pa_{D_i}} \rightarrow \Omega_{D_i}$

where Ω_S *is the cross product of domains in* S*.*

A policy *is a list of decision rules* $\Delta = (\delta_1, \ldots, \delta_m)$

Task: *Find an optimal policy that maximizes the expected utility.*

$$E = \max_{\Delta=(\delta_1,\ldots,\delta_m)} \sum_{x=(x_1,\ldots,x_n)} \prod_i P_i(x) u(x)$$

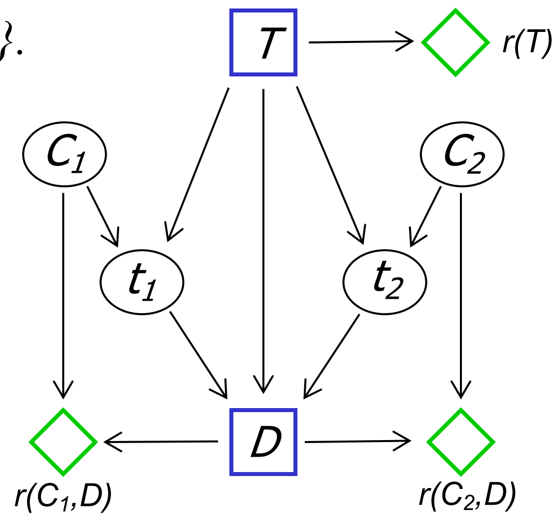
The Car Example (Howard 1976)

A car buyer needs to buy one of two used cars. The buyer can carry out tests with various costs, and then, decide which car to buy.

- T: *Test variable* (t_0, t_1, t_2) (t_1 *test car 1,* t_2 *test car 2)*
- D: *the decision of which car to buy,* $D \in \{buy1, buy2\}$
- C_i: *the quality of car* i, $C_i \in \{q_1, q_2\}$
- t_i: *the outcome of the test on car* i, $t_i \in \{pass, fail, null\}$.
- $r(T)$: *The cost of testing,*
- $r(C_1,D)$, $r(C_2,D)$: *the reward in buying cars 1 and 2.*
- *The utility is:* $r(T) + r(C_1,D) + r(C_2,D)$.

Task: determine decision rules T and D such that:

$$E = \max_{T,D} \sum_{t_2,t_1,C_2,C_1} P(t_2,| C_2, T) P(C_2) P(t_1 | C_1, T) \cdot$$
$$P(C_1)[r(T) + r(C_2, D) + r(C_1, D)]$$



Bucket Elimination for meu (Algorithm Elim-meu-id)

Input: *An Influence diagram* $ID = \{P_1, \ldots, P_n, r_1, \ldots, r_j\}$
Output: *Meu and optimizing policies.*

1. ***Order the variables and partition into buckets.***
2. ***Process buckets from last to first:***

$o = T, t_2, t_2, D, C_2, C_1$

$bucket(C_1)$: $P(C_1), P(t_1|C_1,T), r(C_1,D)$

$bucket(C_2)$: $P(C_2), P(t_2|C_2,T), r(C_2,D)$

$bucket(D)$: $\theta_{C_1}(t_1,T,D), \theta_{C_2}(t_2,T,D)$

$bucket(t_1)$: $\lambda_{C_1}(t_1,T) \quad \theta_D(t_1,t_2,T), \delta(t_1,t_2,T)$

$bucket(t_2)$: $\lambda_{C_2}(t_2,T) \quad \theta_{t_1}(t_2,T)$

$bucket(T)$: $r(T) \quad \lambda_{t_1}(T) \quad \lambda_{t_2}(T) \quad \theta_{t_1}(T)$

θ_T, δ_T

3. ***Forward:*** *Assign values in ordering* d

The Bucket Description

Final buckets: *(λs or Ps) utility components (θ's or r's).*

bucket(C_1): $P(C_1), P(t_1|C_1,T), r(C_1,D)$

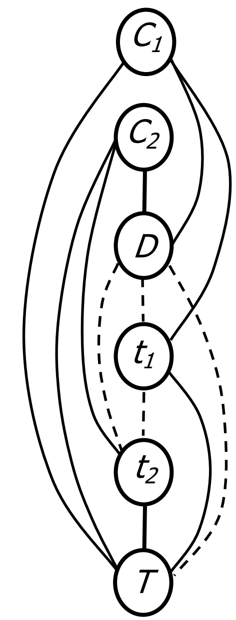
bucket(C_2): $P(C_2), P(t_2|C_2,T), r(C_2,D)$

bucket(D): $\theta_{C_1}(t_1,T,D), \theta_{C_2}(t2,T,D)$

bucket(t_1): $\lambda_{C_1}(t_1,T), \quad \theta_D(t_1,t_2,T)$

bucket(t_2): $\lambda_{C_2}(t_2,T), \quad \theta_{t_1}(t_2,T)$

bucket(T): r(T)

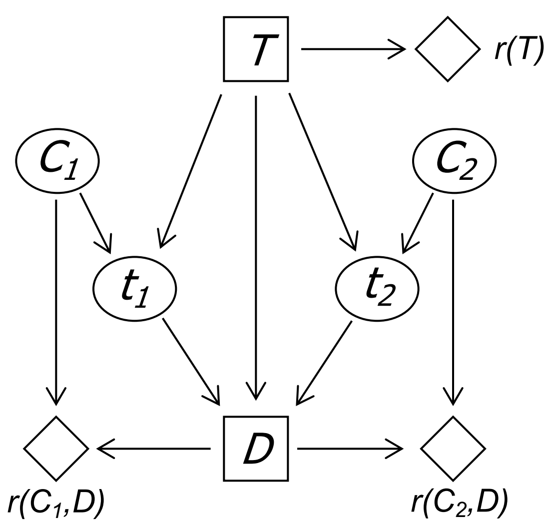


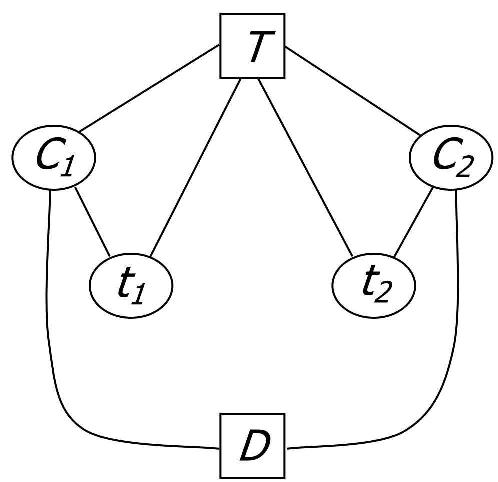
Optimizing policies: δ_T is argmax of θ_T computed in *bucket(T)*, and $\theta_D(t_1,t_2,T)$ in *bucket(t_1)*.

Complexity

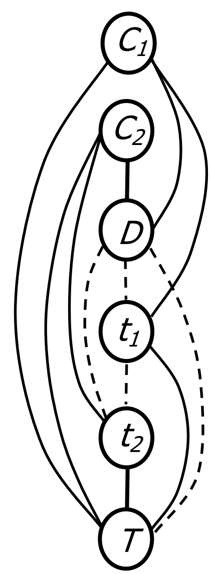
***Theorem**: Algorithm elim-meu-id is time and space $O(n \bullet exp(w^*(o,e))$, where $w^*(o,e)$ is the width along o of the adjusted compounded induced graph.*

*In the **augmented graph** all the parents of chance and value components are connected, Value nodes are deleted.*

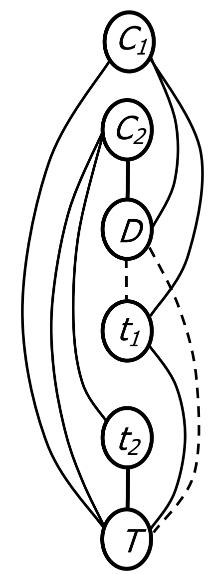




(a)



(b)



(c)

General Graphical Models

Definition 2.2 Graphical model. A *graphical model* $\mathcal{M}$ is a 4-tuple, $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F}, \bigotimes \rangle$, where:

1. $\mathbf{X} = \{X_1, \ldots, X_n\}$ is a finite set of variables;
2. $\mathbf{D} = \{D_1, \ldots, D_n\}$ is the set of their respective finite domains of values;
3. $\mathbf{F} = \{f_1, \ldots, f_r\}$ is a set of positive real-valued discrete functions, defined over scopes of variables $\mathcal{S} = \{S_1, ..., S_r\}$, where $\mathbf{S}_i \subseteq \mathbf{X}$. They are called *local* functions.
4. $\bigotimes$ is a *combination* operator (e.g., $\bigotimes \in \{\prod, \sum, \bowtie\}$ (product, sum, join)). The combination operator can also be defined axiomatically as in [Shenoy, 1992], but for the sake of our discussion we can define it explicitly, by enumeration.

The graphical model represents a *global function* whose scope is $\mathbf{X}$ which is the combination of all its functions: $\bigotimes_{i=1}^{r} f_i$.

General Bucket Elimination

Algorithm General bucket elimination (GBE)

Input: $\mathcal{M} = \langle \mathbf{X}, \mathbf{D}, \mathbf{F}, \otimes \rangle$. $F = \{f_1, ..., f_n\}$ an ordering of the variables, $d = X_1, ..., X_n$; $\mathbf{Y} \subseteq \mathbf{X}$.

Output: A new compiled set of functions from which the query $\Downarrow_Y \otimes_{i=1}^n f_i$ can be derived in linear time.

1. **Initialize:** Generate an ordered partition of the functions into $bucket_1, ..., bucket_n$, where $bucket_i$ contains all the functions whose highest variable in their scope is X_i. An input function in each bucket ψ_i, $\psi_i = \otimes_{i=1}^n f_i$.
2. **Backward:** For $p \leftarrow n$ downto 1, do
for all the functions $\psi_p, \lambda_1, \lambda_2, ..., \lambda_j$ in $bucket_p$, do

- **If** (observed variable) $X_p = x_p$ appears in $bucket_p$, assign $X_p = x_p$ in ψ_p and to each λ_i and put each resulting function in appropriate bucket.
- **else,** (combine and marginalize)
$\lambda_p \leftarrow \Downarrow_{S_p} \psi_p \otimes (\otimes_{i=1}^j \lambda_i)$ and add λ_p to the largest-index variable in $scope(\lambda_p)$.

3. **Return:** all the functions in each bucket.

Theorem 4.23 Correctness and complexity. *Algorithm GBE is sound and complete for its task. Its time and space complexities is exponential in the $w^*(d) + 1$ and $w^*(d)$, respectively, along the order of processing d.*